I. Bayes' Rule

1. Suppose a new cancer test has a 95% chance of correctly identifying that a sick patient has cancer and a 10% chance of incorrectly identifying that a healthy patient has cancer. Assume that 5% of the population has this form of cancer. Compute the following probabilities:

   a) The probability that the test identifies a randomly chosen person as having cancer.

   b) The probability that a person who tests positive for cancer actually has cancer.

   c) The probability that a person who tests negative for cancer does not have cancer.

   d) The probability that the test gives an incorrect result.

   Solution: Let A be the event that a randomly chosen person has cancer, B be the event that the test identifies a randomly chosen person as having cancer. We are given that $P(B|A) = .95$, $P(B|\overline{A}) = .1$, and $P(A) = .05$. Then:
   (a): $P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) = (.95)(.05) + (.10)(.95) = \boxed{.1425}$.
   (b): $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{(.95)(.05)}{.1425} = \boxed{\frac{1}{3}}$.
   (c): $P(\overline{A}|\overline{B}) = \frac{P(\overline{B}|\overline{A})P(\overline{A})}{P(\overline{B})} = \frac{(.9)(.95)}{1-.1425} \approx \boxed{.997}$.
   (d): $P(B|\overline{A})P(\overline{A}) + P(\overline{B}|A)P(A) = (.1)(.95) + (.05)(.05) = \boxed{.0975}$.

2. Suppose a weatherman predicts rainy days correctly with an accuracy of 90% and predicts clear days correctly with an accuracy of 90%. Given that it rains 20% of the time, and the weatherman predicted that it will rain tomorrow, what is the probability that it will actually rain tomorrow?

   Solution: Let A be the event that it rains tomorrow, and let B be the event that the weatherman predicts it will rain tomorrow. We are given that $P(B|A) = .9$, $P(\overline{B}|\overline{A}) = .9$, and $P(A) = .2$. We want to compute $P(A|B)$.
   $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|\overline{A})P(\overline{A})} = \frac{(.9)(.2)}{(.9)(.2)+(.1)(.8)} \approx \boxed{.6923}$

II. Review

1. How many ways can you arrange 10 marbles in a row if 4 are red, 3 are blue, and 3 are green (marbles of the same color are identical)?

   Solution: $\boxed{\binom{10}{4}\binom{6}{3} = \frac{10!}{4!3!3!}}$.

2. How many ways are there to arrange 12 ones and 18 zeros in a line if every one must be immediately followed by a zero?

   Solution: $\boxed{\binom{6+13-1}{13-1} = \binom{18}{12}}$. We group 12 ones and 12 zeros into 12 pairs and the remaining 6 zeros can be filled in to the 13 gaps to produce a valid arrangement of 12 ones and 18 zeros.

3. Show that in a group of 10 people, each of whom is friends with at least one of the others, there are two people with the same number of friends (within the group).

   Solution: People can be friends from 1 to $n-1$ people, so by the pigeonhole principle there are two people with the same number of friends.

4. Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$ using a combinatorial argument and by algebraic manipulation.

   Solution: Both quantities represent the number of ways to choose a team of k people with a captain from a pool of n people (you get the two quantities by either choosing the team first or choosing the captain first).
   Algebraically, we see $k\binom{n}{k} = \frac{n!}{(n-k)!(k-1)!} = n\frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = n\binom{n-1}{k-1}$.

5. How many ways are there to split 6 distinguishable people into 3 distinguishable non-empty teams?

   Solution: $\boxed{3!S(6,3) = 540}$.
   $S(6,3) = \frac{1}{3!}\sum_{i=0}^{2}(-1)^i\binom{3}{i}(3-i)^6 = \frac{1}{6}\left(3^6 - 3\cdot 2^6 + 3\right) = \frac{1}{2}\left(3^5 - 2^6 + 1\right) = 90.$

6. How many non-negative integer solutions are there to the equation $x + y + z = 12$ such that $1 \le x \le 4$?

   Solution: $\boxed{\binom{13}{2} - \binom{9}{2}}$.
   Substituting $x' = x - 1$, we see that we want to count the number of solutions to $x' + y + z = 11$ where $x' \le 3$. There are $\binom{13}{2}$ solutions with no restrictions on x', and $\binom{9}{2}$ solutions when $x' \ge 4$.

7. Let $\{a_n\}_{n\ge 0}$ be the sequence defined by $a_0 = 1$ and $a_{n+1} = 4a_n + 1$. Prove that $a_n = \frac{4^{n+1}-1}{3}$.

   Solution: By induction on n.
   Base Case ($n = 0$): Note $a_0 = 1 = \frac{4^1-1}{3}$.
   Inductive Hypothesis: Assume $a_n = \frac{4^{n+1}-1}{3}$ for some $n \ge 0$. Inudctive Step: Then $a_{n+1} = 4a_n + 1 = 4\frac{4^{n+1}-1}{3} + 1 = \frac{4^{n+2}-4+3}{3} = \frac{4^{n+2}-1}{3}$.
   Thus by inudction $a_n = \frac{4^{n+1}-1}{3}$ for all $n \ge 0$.