MATH 10B with Prof. Stankova
02/28/2019

Discussion on Probability and Independence

1. Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

   (a) The probability of no failures

   > **Solution:** This is the probability of success in each trial, or p^n

   (b) The probability of at least one failure

   > **Solution:** $1 - p^n$

   (c) The probability of at most one failure

   > **Solution:** $p^n + n(1-p)p^{n-1}$, as there are n places to put the failure.

   (d) The probability of at least two failures

   > **Solution:** $1 - p^n - n(1-p)p^{n-1}$

2. Two dice are rolled.

   (a) Are the events that the first die rolled is a 1 and that the sum of the two dice is a 7 independent?

   > **Solution:** $\Pr(A \cap B) = \frac{1}{36}$, as we must roll a 1 and then a 6.
   > Out of the 36 outcomes 6 lead to a sum of 7 and and 6 have 1 as the first number. Therefore $P(A)P(B) = \frac{6}{36}\frac{6}{36} = \frac{1}{36}$, meaning the two events are independent.

   (b) Are the events that the first die rolled is a 1 and that the sum of the two dice is a 6 independent?

   > **Solution:** No. In this case $P(A \cap B) = \frac{1}{36}$ but $P(A)P(B) = \frac{6}{36}\frac{5}{36} \neq \frac{1}{36}$

3. Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on February 1st exceeds 1/2. Assume the probability of being born on any given day is 1/366.

Solution: We look at the complement of this problem. The probability that out of n people none are born on February 1st is $(\frac{365}{366})^n$. The probability that exactly one is born on February 1st is $n\frac{1}{366}(\frac{365}{366})^{n-1}$ (one again there are n places to put the person born on February 1st). Therefore the probability that at least 2 people are born on this day is $1-(\frac{365}{366})^n - n\frac{1}{366}(\frac{365}{366})^{n-1}$. By solving for n we get $n = 614$.

4. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

Solution: Yes they are independent. We can either count scenarios, or if A is the probability that a bit string contains an odd number of ones, and B is the probability of there being a 1 at the beginning, then we have $P(A|B) = P(A) = 1/2$, as regardless of what comes before it, the probability that the number of 1's is odd is determined by the last entry, which is 0 with probability 1/2 and 1 with probability 1/2.

5. If E and F are independent, are E and $\overline{F}$ necessarily independent? Prove or disprove.

Solution: They are independent.

$$\begin{aligned} P(E \cap \overline{F}) &= P(E) - P(E \cap F) \\ &= P(E) - P(E)P(F) \\ &= P(E)(1 - P(F)) \\ &= P(E)P(\overline{F}) \end{aligned}$$

6. Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if

(a) a 0 bit and a 1 bit are equally likely.

Solution: This is just the probability that every bit is 1, so by independence, this is $\frac{1}{2^{10}}$

(b) the probability that a bit is a 1 is 0.6

Solution: $(0.6)^{10}$

(c) the probability that the ith bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \ldots, 10$.

Solution:

$$\prod_{i=1}^{10} \frac{1}{2^i} = \frac{1}{2^{\sum_{i=1}^{10} i}} = \frac{1}{2^{55}}$$