I. Law of large numbers & central limit theorem

1. Let $X_1, X_2, \ldots, X_n$ be Binomial IIDRVs counting the number of heads after flipping a fair coin twice. Let $\overline{X} = \frac{X_1+\cdots+X_n}{n}$ be the average of the first n X_i's, and Z the normalized RV corresponding to $\overline{X}$. For $n = 5$:

(a) List the range $R_{\overline{X}}, \overline{\mu}$, and $\overline{\sigma}$. Compute the p.m.f. $f_{\overline{X}}$ and (optional) sketch its graph.

(b) Which values of $\overline{X}$ are solutions to the inequality $|\overline{X} - \overline{\mu}| > \frac{2}{3}$?

(c) Compute the probability $P(|\overline{X} - \overline{\mu}| > \frac{2}{3})$.

(d) What is the probability $P(|\overline{X} - \overline{\mu}| \leq \frac{2}{3})$?

(e) Rewrite the inequality $|\overline{X} - \overline{\mu}| \leq \frac{2}{3}$ into inequalities of the form $a \leq Z \leq b$ for the normalized RV Z.

(f) Using the standard normal table, approximate $P(a \leq Z \leq b)$ for the two numbers a and b you found above. What is the z-score you used in the table?

(g) What is the relation between $P(|\overline{X} - \overline{\mu}| \leq \frac{2}{3}), P(a \leq Z \leq b)$, and the number you got in part (f)? Are they equal? Approximately equal? Why?

(h) Imagine we have done the same thing for $n = 10$. How would the probability we computed in part (d) change?

II. PDFs and continuous random variables

2. For each of the following, check whether the given function $f(x)$ is a valid PDF, and compute $P(0 < X < 1)$ if so.

(a) $f(x) = 3x^2$ for x between 0 and 1, and 0 otherwise.

(b) $f(x) = xe^{x^2}$ for x between 0 and 1, and 0 otherwise.

(c) $f(x) = \cos x$ for x between 0 and π, and 0 otherwise.

(d) $f(x) = \cos x$ for x between 0 and $\frac{\pi}{2}$, and 0 otherwise.

(e) $f(x) = \frac{1}{x^2}$ for $x \geq 1$, and 0 otherwise.

3. For each of the functions in the previous problem that were not PDFs, how might you turn them into valid PDFs?

4. Let $f(t)$ denote the PDF for the time in minutes it takes to get from Berkeley to San Francisco by BART. Write the probability that your trip takes less than forty minutes as an integral.