I. PDFs and CDFs

1. (a) Call $f(x) = \sin(x) + \frac{1}{2\pi}$ for $0 \leq x \leq 2\pi$ and $f(x) = 0$ otherwise. Verify that $f(x)$ is a valid PDF.
   (b) What is the CDF of this function?
   (c) What is $P(X \leq \pi/2)$?

   Solution:

   In fact, this is not a valid PDF, as this is not always non-negative. For example $\sin(3\pi/2) + \frac{1}{2\pi}$ is negative.

2. (a) Call $f(x) = ae^{-ax}$ for $0 \leq x \leq \infty$ and $f(x) = 0$ otherwise, where $a > 0$ is a constant. Verify that $f(x)$ is a valid PDF.
   (b) What is the CDF of this function?
   (c) What is $P(1 \leq X \leq 2)$?

   Solution:

   (a) We have
   $$\int_0^\infty ae^{-ax} = -e^{-ax}\Big|_0^\infty = 1$$
   and moreover this function is nonnegative.
   (b) The CDF is the antiderivative, so $P(X \leq x) = 0$ for $x \leq 0$ and $1 - e^{-ax}$ for $x \geq 0$.
   (c) This is
   $$\int_1^2 ae^{-ax} = e^{-a} - e^{-2a}$$

3. (a) Call $f(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-z^2}}$ for $-1 \leq x \leq 1$. Verify that $f(x)$ is a valid PDF.
   (b) What is the CDF?
   (c) What is $P(-\frac{1}{\sqrt{2}} \leq X \leq 0)$?

   Solution:

   (a) This function is nonnegative. Then
   $$\int_{-1}^1 \frac{1}{\pi}\frac{1}{\sqrt{1-z^2}} = \frac{1}{\pi}\arcsin(z)\Big|_{-1}^1 = \frac{1}{\pi}(\pi/2 - (-\pi/2)) = 1$$
   (b) This is the antiderivative, so $\frac{1}{\pi}(\arcsin(z) + \pi/2)$ for $-1 \leq z \leq 1$, 0 for $z \leq -1$ and 1 for $z \geq 1$.
   (c) This is
   $$\frac{1}{\pi}(\arcsin(0) - \arcsin(-\frac{1}{\sqrt{2}})) = \frac{1}{\pi}(0 - (-\pi/4)) = \frac{1}{4}$$

4. (a) Call $f(x) = \frac{1}{x}$ for $e \leq x \leq e^2$. Verify that $f(x)$ is a valid PDF.
   (b) What is the CDF?
   (c) What is $P(e^{1.5} \leq X \leq e^{1.75})$?

   Solution:
   (a) This is nonnegative.
   $$\int_e^{e^2} \frac{1}{x} = \ln(x)|_e^{e^2} = \ln(e^2) - \ln e = 1$$
   (b) The CDF is $\ln(z) - 1$
   (c) $\ln(e^{1.75}) - \ln(e^{1.5}) = .25$

5. What PDF has CDF $\sin(x)$ for $0 \leq x \leq \pi/2$?

   Solution We want the antiderivative to be $\sin(x)$, so the PDF must be $\cos(x)$.

6. I flip a fair coin 15 times, counting the number of heads. This is represented by the random variables $X_1, \ldots, X_{15}$. The average of these variables is $\overline{X}$. What is the probability that X_1 and $\overline{X}$ are within one standard error of their expectations? (Calculate exactly!). What is a way to estimate this (do the estimation).

   Solution
   The expectation of X_1 is $1/2$. The standard error is $1/2$, meaning that X has probability 1 of being within one standard error of the expectation.
   The expectation of $\overline{X}$ is also $1/2$. The standard error this time will be $(1/2)/\sqrt{15} \approx .129$, so this is the probability that $P(.371 < \overline{X} < .629)$. As this is an average, this corresponds to getting between $P(5.56 < \sum X_i < 9.44)$. By the binomial distribution this is
   $$\frac{\binom{15}{6} + \binom{15}{7} + \binom{15}{8} + \binom{15}{9}}{2^{15}} \approx .698$$
   By the central limit theorem, this will approximately be equal to the probability that a normal distribution is within 1 standard deviation, which is .683.

7. (Review Question)!
   In the Ancient Greek Attic Calendar, there are 10 months in a year. How many people chosen randomly do we need to ensure that the probability that at least two people are born in the same Attic month is at least 1/2? Assume that birthdays are uniform.

   Solution The complement of this is when everyone is born in different months, which happens with probability $\frac{10}{10}\frac{9}{10}\frac{8}{10}\cdots\frac{10-n+1}{10}$. Therefore the probability that n people are all born on different days is $\frac{P(10,n)}{10^n}$. We can check to see that we need $n = 5$ for this to be at most $1/2$.