Mean, Mode, Variance, Standard Deviation

1. Find the mean and the variance of the random variables with the following PDFs.
   (a) $f(t) = 1$ for $0 \leq t \leq 1$ and $f(t) = 0$ otherwise. **solution**

   Mean:
   $$\int_0^1 x \cdot 1dx = \int_0^1 xdx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$
   Maybe you could have guessed this!

   Variance:
   $$\int_0^1 x^2 \cdot 1dx = \int_0^1 x^2dx = \frac{1}{3}x^2\Big|_0^1 = \frac{1}{3}$$
   so our answer is $\frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$

   (b) $f(x) = \frac{2}{x^3}$ for $1 \leq x \leq \infty$ and $f(x) = 0$ otherwise. **solution** Mean:
   $$\int_1^\infty x \cdot \frac{2}{x^3}dx = \int_1^\infty \frac{2}{x^2}dx = -\frac{2}{x}\Big|_1^\infty = 2$$
   Variance:
   $$\int_1^\infty x^2 \cdot \frac{2}{x^3}dx = \int_1^\infty \frac{2}{x}dx = \ln(x)|_1^\infty = \infty$$
   So in this case we have infinite variance!

   (c) $f(t) = 3t^2$ for $0 \leq t \leq 1$ and $f(t) = 0$ otherwise.

   Mean:
   $$\int_0^1 t \cdot 3t^2dt = \int_0^1 3t^3dt = \frac{3}{4}t^4\Big|_0^1 = \frac{3}{4}$$
   Variance:
   $$\int_0^1 t^2 \cdot 3t^2dt = \int_0^1 3t^4dt = \frac{3}{5}t^5\Big|_0^1 = \frac{3}{5}$$
   so our answer is $\frac{3}{5} - \frac{3^2}{4^2} = \frac{3}{80}$

   (d) $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

   Mean:

   This is a symmetric function so the mean is 0.

   Variance
   $$\int_{-\infty}^\infty x^2 \cdot \frac{1}{2}e^{-|x|}dx = \int_0^\infty x^2e^{-x}dx = -x^2e^{-x}\Big|_0^\infty + 2\int_0^\infty xe^{-x}dx$$
   $$= -2xe^{-x}\Big|_0^\infty + 2\int_0^\infty e^{-x}dx = -2e^{-x}\Big|_0^\infty = 2$$
   As the mean is 0, the variance is 2.

2. What is the mode of the random variable with PDF

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

solution

The mode is where the function reaches its maximum. The easiest way to see this is to graph it. This is a normal distribution, so the maximum is at the mean, μ.

3. What is the standard deviation of the random variable with PDF $2e^{-2t}$ for $0 \leq x \leq \infty$.. Mean:

$$\int_0^\infty t \cdot 2e^{-2t} dt = -te^{-2t}\Big|_0^\infty + \int_0^\infty e^{-2t} dt = 0 - -\frac{1}{2}e^{-2t}\Bigg|_0^\infty = \frac{1}{2}$$

$$\int_0^\infty t^2 \cdot 2e^{-2t} dt = -t^2e^{-2t}\Big|_0^\infty + \int_0^\infty 2te^{-2t} dt = \frac{1}{2}$$

from above. So the variance is $\frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore standard deviation is $\sqrt{\frac{1}{4}} = \frac{1}{2}$

4. What is the standard deviation of the random variable with PDF $\frac{1}{2}e^{-|x|}$?

solution This is the square root of variance, so $\sqrt{2}$