I. Maximum Likelihood Estimation

1. Suppose you flip a coin 100 times and get 30 heads. Estimate the probability p that a single flip of the coin is a head ...

   a) directly (using $\overline{x}$)

      $\hat{p} = \overline{x} = .3$

   b) using maximum likelihood estimation

      $L(p|30) = P(30|p) = \binom{100}{30}p^{30}(1-p)^{70}$. Differentiating with respect to p gives us $\frac{d}{dp}L(p|30) = \binom{100}{30}p^{29}(1-p)^{69}(30(1-p)-70p)$. The zeros of this derivative are $p = 0, 1, \frac{3}{10}$, and plugging these zeros in to $L(p|30)$ gives us the MLE $\hat{p} = \frac{3}{10}$.

   c) Find 90% and 99% confidence intervals for p.

      $\hat{\sigma} = \sqrt{\hat{p}(1-\hat{p})} = .46$. The 90% confidence interval for p is $(\hat{p} - \frac{1.66\hat{\sigma}}{\sqrt{100}}, \hat{p} + \frac{1.66\hat{\sigma}}{\sqrt{100}}) = (.22, .38)$. The 99% confidence interval for p is $(\hat{p} - \frac{2.58\hat{\sigma}}{\sqrt{100}}, \hat{p} + \frac{2.58\hat{\sigma}}{\sqrt{100}}) = (.18, .42)$.

2. Suppose a hospital records the number of critical patients they get per day over the course of 10 days, and get the following data: $10, 4, 3, 7, 5, 8, 2, 11, 12, 8$. Assume that the number of critical patients the hospital receives on any particular day is modeled by a Poisson distribution X with unknown parameter λ. Estimate λ using MLE.

   Let $x_1, \ldots, x_{10}$ be the given data. Then $L(\hat{\lambda}|x_1, \ldots, x_{10}) = P(x_1, \ldots, x_{10}|\hat{\lambda}) = \prod_{i=1}^{10} P(x_i|\hat{\lambda}) = \prod_{i=1}^{10} \frac{x_i^{\hat{\lambda}} e^{-\hat{\lambda}}}{x_i!}$. Thus $\log L(\hat{\lambda}|x_1, \ldots, x_{10}) = -10\hat{\lambda} + \log(\hat{\lambda})(x_1 + \cdots + x_{10}) - \log(x_1!) - \cdots - \log(x_{10}!)$. Taking the derivative of this expression yields $-n + \frac{x_1 + \cdots + x_{10}}{\hat{\lambda}}$. The only zero of this derivative is $\hat{\lambda} = \frac{x_1 + \cdots + x_n}{n} = 7$, which becomes our MLE estimate (you can perform the second derivative test to check that it's a maximum).

3. Suppose X is a geometric random variable with unknown parameter p. You randomly sample X three times and get the values $5, 3, 8$. What is the MLE estimate for p given this data?

   $L(\hat{p}|5, 3, 8) = P(5, 3, 8|\hat{p}) = P(X = 5|\hat{p})P(X = 3|\hat{p})P(X = 8|\hat{p}) = (1-\hat{p})^{16}\hat{p}^3$. Differentiating with respect to $\hat{p}$ yields $(3(1-\hat{p}) - 16\hat{p})(1-\hat{p})^{15}\hat{p}^2$. This has zeros at $\hat{p} = 0, 1, \frac{3}{19}$. The value which maximizes likelihood is $\hat{p} = \frac{3}{19} = \frac{1}{1 + \frac{5+3+8}{3}}$.

4. Suppose X is an exponential random variable with unknown parameter λ. You randomly sample X 5 times and get the values $25, 30, 33, 27, 31$. What is the MLE estimate for λ given this data?

   $L(\hat{\lambda}|x_1, \ldots, x_5) = \hat{\lambda}^5 e^{-\hat{\lambda}(x_1 + \cdots + x_5)}$. Thus $\log L(\hat{\lambda}|x_1, \ldots, x_5) = 5\log(\lambda) - \hat{\lambda}(x_1 + \cdots + x_5)$. Differentiating gives us $\frac{5}{\hat{\lambda}} - (x_1 + \cdots + x_5)$ which has a single 0 at $\hat{\lambda} = \frac{5}{x_1 + \cdots + x_5}$, which is the MLE.

5. Suppose X is a normal random variable with unknown mean and variance μ and σ^2. You randomly sample X 4 times and get the values $3, 4, 6, 7$. What is the MLE estimate for μ and σ^2 given this data?

   Notice $L(\hat{\mu}, \hat{\sigma}|x_1, \ldots, x_4) = \prod_{i=1}^{4} \frac{1}{\hat{\sigma}\sqrt{2\pi}} e^{-\frac{(x_i - \hat{\mu})^2}{2\hat{\sigma}^2}}$. Let's assume $\hat{\sigma}$ is a constant and find the $\hat{\mu}$ which maximizes the likelihood. We'll look at the log likelihood here:

   $$\log L(\hat{\mu}, \hat{\sigma}|x_1, \ldots, x_4) = \sum_{i=1}^{4} -\log(\hat{\sigma}\sqrt{2\pi}) - \frac{(x_i - \hat{\mu})^2}{2\hat{\sigma}^2}$$

Setting the derivative with respect to $\hat{\mu}$ equal to 0, we get $\sum_{i=1}^{4} \frac{2(x_i-\hat{\mu})}{2\hat{\sigma}^2} = 0$, which has one solution - $\hat{\mu} = \overline{x}$. Now differentiating with respect to $\hat{\sigma}$ (after replacing $\hat{\mu}$ with $\overline{x}$), we get:

$$0 = \frac{-4}{\hat{\sigma}} + \sum_{i=1}^{4} \frac{(x_i - \overline{x})^2}{\hat{\sigma}^3}$$

So $\hat{\sigma}^2 = \frac{1}{4} \sum_{i=1}^{4} (x_i - \overline{x})^2$.

6. For each of the above problems, determine whether the MLE estimate you obtained was biased or unbiased.

   The MLE estimate is unbiased except in the estimate for p in the Geometric distribution, the estimate for λ in the exponential distribution, and the estimate for σ^2 in the normal distribution.