1 Hypothesis testing

1. Let X be a measurement of a patient's blood pressure, which follows the normal distribution with $\sigma = 5$ mmHg center around the true blood pressure. Let the null hypothesis be that the patient's blood pressure is $\mu = 120$, and the alternative hypothesis $H_1: \mu > 120$ with significance level $\alpha = 0.05$? Supposed that the patient's blood pressure is measure to be 135 mmHg.

   (a) Is it a one-sided test?
   (b) Calculate the p-value.
   (c) Calculate critical value and find the rejection region.
   (d) Draw a conclusion.

   Solution:
   i. It is a one sided test.
   ii. Assume the null is true, $z = \frac{X-\mu}{\sigma} = \frac{135-120}{5} = 3$. Looking up the table, Three SD corresponds to an area of 0.4987. Since it is a one-sided test, the p-value is $0.5 - 0.4987 \approx 0.0013$.
   iii. Let k be the critical value. Then, assuming the null is true, $P(X > k) = 0.05 \implies P(\frac{X-120}{5} > \frac{k-120}{5}) = P(Z > \frac{k-120}{5}) = 0.05$. Thus, $\frac{k-120}{5} > 1.65 \implies k > 128.25$, and the rejection region is $[128.25, \infty)$.
   iv. Since the p-value is $0.0013 < 0.05$ or $X = 135 > 128.25$, we can reject the null and conclude that $\mu > 120$.

2. Let X be a measurement of a patient's blood pressure, which follows the normal distribution with $\sigma = 5$ mmHg center around the true blood pressure. Let the null hypothesis be that the patient's blood pressure is $\mu = 120$, and the alternative hypothesis $H_1: \mu \neq 120$ with significance level $\alpha = 0.05$? Supposed that the patient's blood pressure is measure to be 130 mmHg.

   (a) Is it a one-sided test?
   (b) Calculate the p-value.
   (c) Draw a conclusion.

   Solution:
   i. It is a two sided test.
   ii. Assume the null is true, $z = \frac{X-\mu}{\sigma} = \frac{130-120}{5} = 2$. Looking up the table, Three SD corresponds to an area of 0.4772. Since it is a one-sided test, the p-value is $2(0.5 - 0.4772) \approx 0.0456$.
   iii. Since the p-value is $0.0456 < 0.05$, we can reject the null and conclude that $\mu \neq 120$.

3. You have a coin that you suspect is biased toward coming up heads. You let p be the probability that the coin will give heads on any one flip. Can you reject the null hypothesis $H_0: p = \frac{1}{2}$ in favor of the alternative hypothesis $H_1: p > \frac{1}{2}$ with significance level $\alpha = 0.05$?

(a) you get 7 heads in 8 flips

Solution: Assuming the null is true, the $p{-}value$ in this case is $\binom{7}{8}(\frac{1}{2})^7(1-\frac{1}{2})+\binom{8}{8}(\frac{1}{2})^8 = 0.03515625 < 0.05$. Thus, reject the null.

(b) you get 6 heads in 8 flips

Solution: Assuming the null is true, the $p - value$ in this case is $\binom{6}{8}(\frac{1}{2})^6(1-\frac{1}{2})^2 + \binom{7}{8}(\frac{1}{2})^7(1-\frac{1}{2}) + \binom{8}{8}(\frac{1}{2})^8 = 0.14453125 > 0.05$. Thus, not sufficient evidence to reject the null.

4. You have a coin that you suspect is biased toward coming up heads. You let p be the probability that the coin will give heads on any one flip.

   (a) Explain why, if you flip the coin three times, then no matter how the three flips turn out, you will NOT have enough evidence to reject the null hypothesis "$p = \frac{1}{2}$ " in favor of the alternative hypothesis "$p > \frac{1}{2}$ at significance level $\alpha = 0.05$. Hint: calclute the $p - values$ if you have $3, 2, 1, 0$ heads and compare with α.

Solution: In general, when applying a hypothesis test to a data set, the corresponding p-value is the probability of the data to be as strange/extreme for the null hypothesis to be true. For any significance level α, if the p-value for the data found is less than α, then there is enough evidence to reject the null hypothesis at significance level α.

For this data set, $\alpha = 0.05$ and the null hypothesis H_0 is "$p = \dfrac{1}{2}$" and the alternative hypothesis H_a is "$p > \dfrac{1}{2}$". Now suppose one flips the coin n times and k of those flips are heads. For the null hypothesis $p = \dfrac{1}{2}$ to be true and the alternative hypothesis $p > \dfrac{1}{2}$ to be false, at least k heads should be obtained. In other words, if we define "X" to be "the number of heads you get after n flips" and we actually get k flips, the p-value is $P(X \geq k)$, and we have enough data to reject H_0 at significance level $\alpha = 0.05$ if:

$$P(X \geq k) \leq 0.05.$$

If $n = 3$, the PMF of X for H_0 to be true is:

$$P(X = k) = \binom{3}{k}\left(\frac{1}{2}\right)^k\left(1 - \frac{1}{2}\right)^{3-k} = \binom{3}{k} \cdot \frac{1}{2^k} = \frac{1}{8} \cdot \binom{3}{k}.$$

So if one flips the coin 3 times and get all heads, the p-value is:

$$P(X \geq 3) = P(X = 3) = \frac{1}{8} \cdot \binom{3}{3} = \frac{1}{8} > 0.05.$$

If one flips the coin 3 times and get less than 3 heads, which is less extreme than getting 3 heads, the p-value will be greater than the p-value of getting 3 heads. So no matter how many heads will be obtained, there will not be enough evidence to reject the statement "$p = \dfrac{1}{2}$" with significance $\alpha = 0.05$.

(b) What is the smallest number of times you need to flip the coin so that having all of those flips turn out to be heads is enough evidence to reject the null hypothesis "$p = \frac{1}{2}$ " in favor of the alternative hypothesis "$p > \frac{1}{2}$ at significance level $\alpha = 0.05$.

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Solution: If H_0 were true (i.e. $p = \frac{1}{2}$), for H_0 to be true and H_a to be false, there is nothing more extreme than flipping a coin n times and getting n heads. So if one flips a coin n times and get n heads, the p-value for your data is:

$$P(X \geq n) = P(X = n) = \binom{n}{n}\left(\frac{1}{2}\right)^n\left(1 - \frac{1}{2}\right)^{n-n} = 1 \cdot \frac{1}{2^n} \cdot 1 = \frac{1}{2^n}.$$

We would like to know the smallest positive integer n for which $\frac{1}{2^n} < 0.05$. If $n = 4$,

$$\frac{1}{2^n} = \frac{1}{16} > \frac{1}{20} = 0.05$$

but if $n = 5$,

$$\frac{1}{2^n} = \frac{1}{32} < 0.05$$

So $\boxed{5}$ is the minimum number of flips one needs to reject "$p = \frac{1}{2}$"" in favor of "$p > \frac{1}{2}$" at a significance level of $\alpha = 0.05$.