Worksheet 28

.5cm

1. How many functions are there from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3\}$ such that $f(3)$ is odd?

   I have 5 slots, one for each element of my domain. Each has value 3 except for 3, which only has 2 options. So my answer is $3^4 \cdot 2 = 162$.

2. How many solutions are there to $x_1 + x_2 + x_3 \leq 20$ such that $3 \leq x_1 \leq 6, 1 \leq x_2, 3 \leq x_3$?
   We set $y_1 = x_1 - 3, y_2 = x_2 - 1, y_3 = x_3 - 3$. Moreover we add a variable to represent the remainder below 20 x_4. We then get $y_1 + y_2 + y_3 + x_4 \leq 13$, $0 \leq y_1, y_2, y_3, x_4$, $y_1 \leq 3$. Therefore we do two balls and urns problems, one where we ignore the requirement on y_1 and another where we subtract the complement $y_1 \geq 4$. This gives us

$$\binom{13+4-1}{3} - \binom{9+4-1}{3} = 340$$

3. Prove that the number of diagonals of a convex n-gon is $n(n-3)/2$. Test this out for $n = 3$ and $n = 4$ to make sure you believe this.

   Induct! Base step is for a triangle. There are zero diagonals of a triangle.

   Inductive hypothesis: assume the statement is true for an n-gon for some n.

   Inductive step: Take an $n + 1$-gon. By connecting two vertices distance 2 away, we create an n-gon. The number of diagonals that are not contained in this n-gon are the diagonals of the point between the two vertices, and the connection of the 2 vertices. Therefore the total number of diagonals is

$$\frac{n(n-3)}{2} + n - 3 + 1 = \frac{n^2 - n - 2}{2} = \frac{(n+1)(n-2)}{2}$$

   Therefore by mathematical induction, we have proved the statement.

4. How many integers from 50 to 100 (inclusive) are divisible by 7 but not 4?

   Number of integers from 1 to 100 divisible by 7 is $\lfloor \frac{100}{7} \rfloor = 14$. The number of integers from 1 to 49 that are divisible by 7 is 6. Therefore the number of integers divisible by 7 between 50 and 100 is 7. Similarly, we find that the number of integers from 50 to 100 divisible by 28 is 2. Therefore there are 7-2=5 integers divisible by 7 but not 4 between 50 and 100.

5. How many ways can you line up 10 men and 5 women such that no two women stand next to each other?
   There are 10! ways to line up the men. Then we place the women, for which there are 11 distinct places to put them between the men. Therefore the answer is $10!P(11, 5)$

6. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

   Pigeonhole! For the 101 people (pigeons) place them into holes based on the longest decreasing height sequence starting at that person. This cannot have length more than 10, otherwise we're done. Therefore there must be 11 people in one hole. However, this means that the 11 people each form a different decreasing subsequence, meaning they form an increasing subsequence.

7. When a test for steroids is given to soccer players, 98% of the players taking steroids test positive and 12% of the players not taking steroids test positive. Suppose that 5% of soccer players take steroids. What is the probability that a soccer player who tests positive takes steroids?

   Bayes Theorem (or a tree diagram).

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}$$

   , where A is taking steroids and B is testing positive. Plugging in numbers we get 30%

8. Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped. What is the expected value of X_n? What is the variance of X_n?

   The expectation is $E(T - H) = E(T) - E(H) = 0$

   Variance is

$$\text{Var}(T - H)\text{Var}(T - (n - T)) = \text{Var}(2T - n) = \text{Var}(2T) = 4\frac{n}{4} = n$$

   Remember that this is a binomial distribution, which is how we get variance.