Green's function techniques for random regular graphs.

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Abstract

Universality is the phenomenon that many random matrix models share the same spectral statistics, even when their entries come from very different distributions. Recent breakthroughs show that this principle reaches far beyond classical matrix ensembles: although random regular graphs are constructed from far fewer independent sources of randomness, and although the entries of their adjacency matrices are dependent, the distribution of the spectral edge agrees with that of the Gaussian Orthogonal Ensemble. This reveals a form of fully chaotic behavior in a highly constrained sparse model and proves the existence of optimally expanding graphs of every degree, resolving a longstanding open question. In this mini-course, we will explain key ideas behind this surprising analogy and introduce the main tools connecting sparse random graphs with random matrix theory.

Preface

These notes are created for the mini-course as as part of the Clay Mathematics Institue-Heilbronn Institute for Mathematical Research Summer School on Random Geometries and Random Matrices. I am grateful to Emma Bailey and Laura Monk for giving me the opportunity to give a mini-course. The goal of these notes is to introduce some of the ideas needed to find the distribution of the edge eigenvalues of the random regular graph. As some of the ideas necessary are quite involved, I have limited myself to a few key fundamental insights necessary to solve the question. For a description of the entire proof, I recommend Jiaoyang Huang and Hong-Tzer Yau's lecture notes [3].

1 Introduction

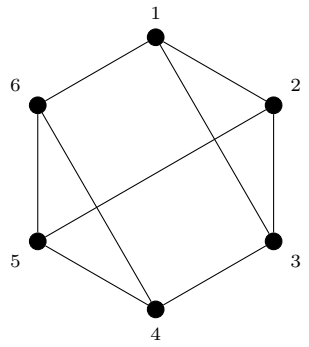
1.1 The goals of this course

Our goal is to understand the extreme eigenvalues of random regular graphs and explain why, with positive probability, they have optimal spectral expansion. A graph is a set of vertices connected by edges. In these notes, the vertex set will always be the set $[\![N]\!] := \{1, 2, 3, \dots, N\}$. Therefore, we can write the graph as $\mathcal{G} = ([\![N]\!], E)$, where E is the set of edges.

There is a rich spectral theory associated with graphs, and we will focus now on graphs conditioned so that every vertex neighbors exactly d other vertices, for some constant $d \geqslant 3$. Such a graph is called *d-regular*. We will often identify a d-regular graph with its normalized adjacency operator $H_{\mathcal{G}} := A_{\mathcal{G}}/\sqrt{d-1}$ (the reason for this normalization will be clear later), where $A_{\mathcal{G}}$ is the adjacency matrix. To be precise, $H_{\mathcal{G}}$ is an $N \times N$ matrix where for each $1 \leqslant u, v \leqslant N$,

$$(H_{\mathcal{G}})_{uv} = \begin{cases} 1/\sqrt{d-1} & (u,v) \in E \\ 0 & (u,v) \notin E. \end{cases}$$

Note $H_{\mathcal{G}}$ is symmetric. Therefore, it has real eigenvalues. Moreover, its largest eigenvalue is always $d/\sqrt{d-1}$, which corresponds to the all ones eigenvector. More generally, we can write its eigenvalues as $d/\sqrt{d-1} = \lambda_1(\mathcal{G}) \geqslant \lambda_2(\mathcal{G}) \geqslant \cdots \geqslant \lambda_N(\mathcal{G}) \geqslant -d/\sqrt{d-1}$.



$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 1: A 3-regular graph on six vertices and its normalized adjacency operator $H = A/\sqrt{2}$.

Similar to surfaces, the spectral theory of graphs is a richly studied subject. The purpose of this lecture series is to introduce the techniques used to find the distribution of the *spectral gap* over a randomly selected regular graph. In this work, this key quantity is the largest nontrivial eigenvalue in absolute value,

$$\max\{\lambda_2(\mathcal{G}), -\lambda_N(\mathcal{G})\},$$

which controls the expansion of the graph and the convergence behavior of random walk. There is a natural bound on the size of these eigenvalues based just on the degree.

Proposition 1.1 (Alon-Boppana 80's)**.** *For $\mathcal{G}$ an N vertex, d-regular graph for $d \geqslant 2$ fixed, $\lambda_2(\mathcal{G}) \geqslant 2 - o_N(1)$.*

The size of this eigenvalue controls the spectral expansion. Therefore, people became interested in showing this bound in the other direction. Friedman showed an almost optimal bound for a randomly selected graph.

Definition 1.2. *We define the* random regular graph ensemble $\mathcal{G}(N, d)$ *by taking the set of all N vertex, d-regular graphs, and selecting one uniformly at random.*

Theorem 1.3 (Friedman)**.** *For any fixed $d \geqslant 3$, with probability $1 - o_N(1)$, $\max\{\lambda_2(\mathcal{G}), -\lambda_N(\mathcal{G})\} \leqslant 2 + o_N(1)$.*

We care about $\lambda_N(\mathcal{G})$ as well as this governs the mixing time of the random walk.

A natural question is whether the $o_N(1)$ error can be removed. This is known as a Ramanujan graph.

Definition 1.4. *A graph is called* Ramanujan *if* $\max\{\lambda_2(\mathcal{G}), -\lambda_N(\mathcal{G})\} \leqslant 2$.

Such a graph is an optimal expander, as we have removed this error term, and we cannot do any better as the Alon-Boppana bound is tight. For many applications, Friedman's theorem is sufficient; however, for example, there are certain algorithms where the dependence on this error term blows up at a rate of $\exp(N\sqrt{(\lambda_2 - 2)})$. Therefore, it is advantageous to create graphs that actually satisfy this bound. Previous constructions of Ramanujan graphs come from number theory and rely on highly structured algebraic objects. For example, showing that there is a Ramanujan graph of degree $d = 7$ was an open problem, and even for other degrees, it was unclear whether every N had a Ramanujan graph (here we ignore the bipartite version of this problem, for which the existence result had previously been solved [4,5]). In fact, based on computer tests, a random regular graph satisfied this Ramanujan bound sometimes.

Definition 1.5 (GOE matrix)**.** *An $N \times N$ random symmetric matrix Z is said to be drawn from the Gaussian Orthogonal Ensemble, written $Z \sim \mathrm{GOE}_N$, if its entries are independent up to the symmetry*

constraint $Z_{ij} = Z_{ji}$ *and satisfy*

$$Z_{ij} \sim \mathcal{N}\left(0, \frac{1}{N}\right), \qquad i < j,$$

and

$$Z_{ii} \sim \mathcal{N}\left(0, \frac{2}{N}\right).$$

Definition 1.6 (Tracy–Widom TW_1 distribution)**.** *The Tracy–Widom* TW_1 *distribution is the limiting law of the rescaled largest eigenvalue of a GOE matrix. More precisely, if* $Z_N \sim \mathrm{GOE}_N$ *and* $\lambda_{\max}(Z_N)$ *denotes its largest eigenvalue, then*

$$N^{2/3}\left(\lambda_1(Z_N) - 2\right) \Longrightarrow \mathrm{TW}_1$$

as $N \to \infty$*.*

This allows us to show the main theorem, which proves a conjecture of Sarnak [7] and Miller, Novikoff, and Sabelli [6].

Theorem 1.7 (Theorem 1.2 of [1])**.** *Define* $\mathcal{A} := \frac{d(d-1)}{(d-2)^2}$*. For* $d \geqslant 3$*, the following is true for any fixed* $s \in \mathbb{R}$*,* $\mathcal{G} \sim \mathcal{G}(N, d)$ *and* $X \sim TW_1$*,*

$$\mathbb{P}((\mathcal{A}N)^{2/3}(\lambda_2(\mathcal{G}) - 2) \leqslant s) = \mathbb{P}(X \leqslant s) + o_N(1).$$

The same is true for $-\lambda_N(\mathcal{G})$*, and the two random variables are independent.*

As these are independent Tracy Widom distributions, as $\mathbb{P}(TW_1 \leqslant 0) \approx 0.83...$ the following is true.

Corollary 1.8. *For any fixed* $d \geqslant 3$ *and sufficiently large* N*,*

$$\mathbb{P}\left(\max\{\lambda_2(\mathcal{G}), -\lambda_N(\mathcal{G})\} \leqslant 2\right) \approx 0.69.$$

This means that most, in the sense that more than 50% graphs are Ramanujan. Our goal will be to give the techniques necessary to show the above theorem.

1.2 Green's function analysis

The main technique that I want to discuss with you all is how to perform analysis of the *Green's function* in this model. For any self-adjoint operator H and $z : \mathrm{Im}[z] > 0$, we take the Green's function $G(z, H) := (H - z\mathbb{I})^{-1}$, and its normalized trace the *Stieltjes transform* $m_N(z) := \frac{1}{N}\operatorname{Tr}[G(z)]$. This gives us information about the eigenvalues, because if we split $z = E + \mathrm{i}\eta$, $\mathrm{Im}[m_N(z)] = \frac{\eta}{N}\sum_{i \in [\![N]\!]} \frac{1}{|\lambda_i - E|^2 + \eta^2}$. Therefore, if we can get sharp information about the Green's function when $|E| \approx 2$ and η is very small, we can track the distribution around 2 and -2.

Thus, sufficiently precise control of the Green's function gives detailed information about the eigenvalue distribution. If you have not seen this used before, a question may be why to choose the Green's function, rather than any other family of analytic functions. One large reason is because there are many formulae associated with the inverses of matrices, which allow us to manipulate them.

An important example of these manipulations, is one often used in random matrix theory, which is the *Schur complement formula*. This tells us how to decompose an inverse in terms of the original matrix. Let

$$M = \begin{bmatrix} A & B^* \\ B & D \end{bmatrix}.$$

Then, assuming D and $A - B^*D^{-1}B$ are invertible,

$$M^{-1} = \begin{bmatrix} (A - B^*D^{-1}B)^{-1} & -(A - B^*D^{-1}B)^{-1}B^*D^{-1} \\ -D^{-1}B(A - B^*D^{-1}B)^{-1} & D^{-1} + D^{-1}B(A - B^*D^{-1}B)^{-1}B^*D^{-1} \end{bmatrix}.$$

To apply this to the Green's function, for any $\mathbb{V} \subseteq V$, we define $G^{(\mathbb{V})}$ to be the Green's function of the graph

$$\mathcal{G}^{(\mathbb{V})} := \mathcal{G} \setminus \mathbb{V}.$$

Writing $H - z$ in block form with respect to $\mathbb{T} \sqcup \mathbb{T}^c$ for some set of vertices $\mathbb{T}$,

$$H - z = \begin{pmatrix} H|_{\mathbb{T}} - z & B^* \\ B & H|_{\mathbb{T}^c} - z \end{pmatrix},$$

we obtain

$$\begin{aligned} G|_{\mathbb{T}} &= \left(H|_{\mathbb{T}} - z - B^* G^{(\mathbb{T})} B\right)^{-1}, \\ G^{(\mathbb{T})} &= G|_{\mathbb{T}^c} - G|_{\mathbb{T}^c\mathbb{T}} (G|_{\mathbb{T}})^{-1} G|_{\mathbb{T}\mathbb{T}^c}, \\ G|_{\mathbb{T}\mathbb{T}^c} &= -G|_{\mathbb{T}} B^* G^{(\mathbb{T})}. \end{aligned}$$

Our goal will be to show that the Green's function on a random regular graph approximates that of the d-regular infinite tree $\mathcal{X}$. The reason is that locally, most neighborhoods are tree-like.

Definition 1.9. *Fix $d \geqslant 3$ and $0 < \mathfrak{c} < 1$, $\mathfrak{R} = (\mathfrak{c}/4) \log_{d-1} N$. We define $\mathcal{B}_{\mathfrak{R}}(o, \mathcal{G})$ to be the ball of radius $\mathfrak{R}$ around o in $\mathcal{G}$. We define the event $\overline{\Omega}$, where the following occur:*

1. *The number of vertices that do not have a tree neighborhood of radius $\mathfrak{R}$ is at most $N^{\mathfrak{c}}$.*
2. *The radius $\mathfrak{R}$ neighborhood of each vertex has an excess (i.e., the number of independent cycles) of at most 1.*

The following proposition states that $\overline{\Omega}$ holds with high probability.

Proposition 1.10 ([2, Proposition 2.1])**.** *$\overline{\Omega}$ occurs with probability $1 - \mathrm{O}(N^{-(1-\mathfrak{c})})$.*

Proof Sketch. The general idea is that having a cycle is quite unlikely. For example, to find the probability of having a triangle, the following idea basically works. There are $d(d-1)$ paths of depth 2. The probability that any such path hits one of the original neighbors is (d/N). Therefore, the expected number of such paths is like $d(d-1) \cdot d/N$. So for any fixed vertex, the probability we see one is low. Nevertheless, the expected number of these that exist in the larger graph is at least constant.

The full argument extends this counting idea to all cycles of length at most $2\mathfrak{R}$, and then to pairs of cycles in the same radius-$\mathfrak{R}$ ball. Since $\mathfrak{R}$ is logarithmic with sufficiently small constant, the expected number of bad vertices beyond the allowed threshold is small. □

The Stieltjes transform of the infinite regular tree satisfies a well-behaved *self-consistent equation*. Specifically, we consider the unique solutions in the upper half plane to the following equations.

$$m_d(z) = (-z - \frac{d}{d-1} m_{sc}(z))^{-1}, \qquad m_{sc}(z) = (-z - m_{sc}(z))^{-1}.$$

We then have the following.

Proposition 1.11. *Let $\mathcal{X}$ be the infinite d-regular tree, and let $H = A_{\mathcal{X}}/\sqrt{d-1}$. For all $z \in \mathbb{C}_+$,*

$$G_{xx}(z) = m_d(z).$$

Moreover, if $x \sim y$, then

$$G^{(x)}_{yy}(z) = m_{sc}(z).$$

Proof. Fix an oriented edge $x \sim y$. Applying the Schur complement formula in the graph $\mathcal{X} \setminus \{x\}$ gives

$$G_{yy}^{(x)} = \frac{-1}{z + \frac{1}{d-1}\sum_{k,\ell \in \partial y \setminus \{x\}} G_{k\ell}^{(xy)}}.$$

After deleting both x and y, the branches rooted at the vertices $k \in \partial y \setminus \{x\}$ are disconnected. Hence the off-diagonal terms vanish, and

$$G_{yy}^{(x)} = \frac{-1}{z + \frac{1}{d-1}\sum_{k \in \partial y \setminus \{x\}} G_{kk}^{(xy)}}.$$

By homogeneity, each term $G_{kk}^{(xy)}$ is equal to $G_{yy}^{(x)}$. Therefore $m := G_{yy}^{(x)}$ satisfies

$$m = \frac{-1}{z+m}.$$

The unique solution with $\operatorname{Im} m > 0$ is $m_{sc}(z)$.

Applying the Schur complement formula at x in the full tree gives

$$G_{xx} = \frac{-1}{z + \frac{d}{d-1} m_{sc}(z)} = m_d(z).$$

□

Exercise 1.12. *Use the formula $(H-z)G = I$ to show that, more generally, $G_{xy} = m_d(z)\left(-\frac{m_{sc}(z)}{\sqrt{d-1}}\right)^{\operatorname{dist}(x,y)}$*

There are two main takeaways from this. First, to find the Kesten-McKay Green's function, it is easier to pass to the Stieltjes transform of the semicircle law, and to find the semicircle law, we can find the self-consistent equation.

In order to properly compare the Stieltjes transform of the finite graph with that of the infinite graph, we will go through a similar process. Recall that we obtained the $(d-1)$-ary tree by deleting a vertex from the infinite d-regular tree and then looking at the Green's function at one of its neighbors. We will do the same thing in an averaged sense for finite graphs. Namely, we will deduce $m_N \approx m_d$ by considering a quantity Q that approximates the semicircle Stieltjes transform m_{sc}, and then showing that $Q \approx m_{sc}$ implies $m_N \approx m_d$.

To this end, let $\vec{E}$ denote the set of directed edges of $\mathcal{G}$, so that each undirected edge $\{o, i\}$ contributes the two directed edges (o, i) and (i, o). We then define

$$Q(\mathcal{G}, z) := \frac{1}{Nd} \sum_{(o,i) \in \vec{E}} G_{oo}^{(i)}. \tag{1.1}$$

Our general goal will be to show that $Q(z) \approx m_{sc}(z)$. Since m_{sc} is characterized by the self-consistent equation

$$m_{sc} = (-z - m_{sc})^{-1},$$

it is natural to prove an approximate version of $Q \approx (-z-Q)^{-1}$. The reason this should be true is local: after deleting a neighbor i of o, the Green's function entry $G_{oo}^{(i)}$ sees, to first order, a rooted $(d-1)$-ary tree. Since random regular graphs are locally tree-like, Q should behave like m_{sc}.

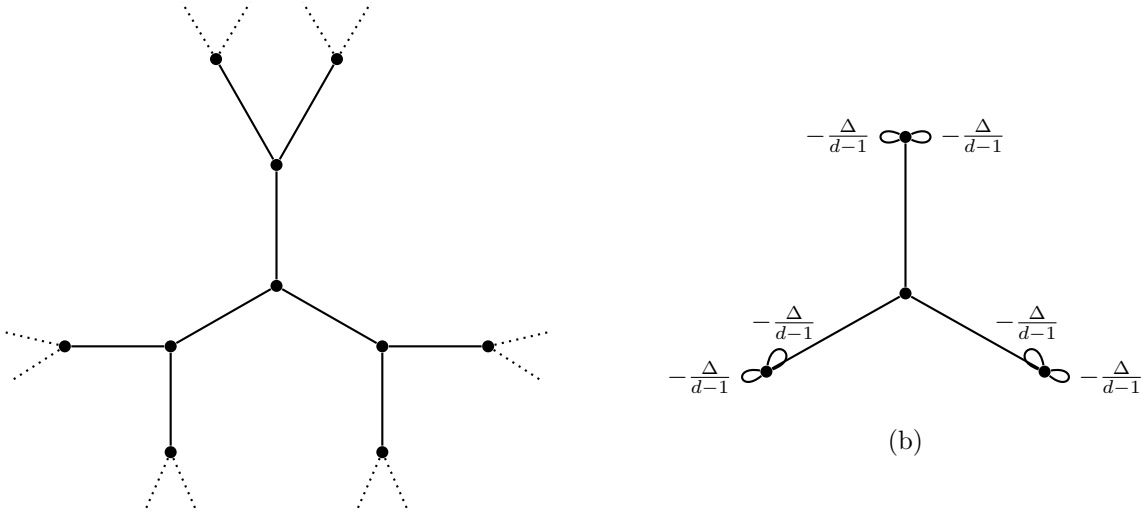
1.3 Generalized self-consistent equations

We now turn the tree recursion from the previous section into a finite-depth approximation. The idea is to replace the part of the graph outside a finite neighborhood by a self-energy parameter $\Delta(z)$. For a subgraph $\mathcal{H} \subset \mathcal{G}$, define the boundary deficit

$$g_{\mathcal{H}}(v) := \deg_{\mathcal{G}}(v) - \deg_{\mathcal{H}}(v), \qquad v \in V(\mathcal{H}).$$

We want to show that our Green's function is reasonably approximated after passing to this neighborhood. Therefore, we define the Green's function of the graph after *extending* with weight function $\Delta(z)$ as

$$G(\mathrm{Ext}(\mathcal{H}, \Delta(z)), z) := \left(-z + H|_{\mathcal{H}} - \frac{\Delta}{d-1} \sum_{v \in \mathcal{H}} g_{\mathcal{H}}(v) e_v e_v^* \right)^{-1}.$$



(a) (b)

Figure 2: Left: a depth-2 neighborhood of the infinite 3-regular tree, with dotted edges indicating continuation to infinity. Right: the central vertex and its three neighbors, where each neighbor carries two $-\frac{\Delta}{d-1}$-loops.

In order to show this and create a self-consistent equation that shows that $Q \approx m_{sc}$, we define $\mathcal{Y}$ to be the infinite $(d-1)$-ary tree with root o and $\mathcal{X}$ to be the infinite d-regular tree with root o. For any function $\Delta(z) : \mathbb{C}^+ \to \mathbb{C}^+$ and $\ell \in \mathbb{Z}_{\geqslant 0}$, define

$$X_\ell(\Delta, z) := G_{oo}(\mathrm{Ext}(\mathcal{B}_\ell(o, \mathcal{X}), \Delta(z)), z), \quad Y_\ell(\Delta, z) := G_{oo}(\mathrm{Ext}(\mathcal{B}_\ell(o, \mathcal{Y}), \Delta(z)), z). \tag{1.2}$$

Note that $X_\ell(m_{sc}(z)) = m_d(z)$ and $Y_\ell(m_{sc}(z)) = m_{sc}(z)$.

Moreover, we can use the following expansion.

Exercise 1.13. *We have that the following Taylor expansions.*

$$X_\ell(\Delta(z), z) - m_d(z) = \frac{d}{d-1} m_d(z)^2 m_{sc}(z)^{2\ell} (\Delta(z) - m_{sc}(z)) + \mathrm{O}(\ell |\Delta(z) - m_{sc}(z)|^2),$$

and

$$Y_\ell(\Delta(z), z) - m_{sc}(z) = m_{sc}(z)^{2\ell+2} (\Delta(z) - m_{sc}(z)) + \mathrm{O}(\ell |\Delta(z) - m_{sc}(z)|^2),$$

Proposition 1.14. *With probability at least $1 - N^{-1+o_N(1)}$, the following is true of $\mathcal{G} \sim \mathcal{G}(N, d)$. Fix any $0 < \mathfrak{c} < .01$, and $\ell := \mathfrak{c} \log_{d-1} N$. For every z such that $\mathrm{Im}[z] \geqslant N^{-1+o_N(1)}$ and $|z| \leqslant 2.01$, we have that*

$$|m_N(z) - X_\ell(Q)|, |Q - Y_\ell(Q)| \leqslant N^{o_N(1)}(d-1)^{2\ell} \frac{\mathrm{Im}[m_d(z)]}{N\eta}.$$

This is enough to deduce a high probability bound.

Corollary 1.15. *With high probability, the second largest eigenvalue satisfies $\lambda_2 - 2 \leqslant N^{-2/3+o_N(1)}$.*

Proof. Recall we set $\ell = \mathfrak{c} \log_{d-1} N$ for some $\mathfrak{c} > 0$. We will use the above equation to show that there is no eigenvalue at $2 + \kappa$ for $\kappa = N^{-2/3+4\mathfrak{c}}$. To do this, we compare the Green's functions in this region. $z = 2 + \kappa + \mathrm{i}\eta$, for η that we set later. In order to compare the random quantity $m_N(z, \mathcal{G})$ with the deterministic quantity, we use the following estimations, which can be proven as a simple exercise.

$$\mathrm{Im}[m_d(z)] \asymp \mathrm{Im}[m_{sc}(z)] \asymp \begin{cases} \frac{\eta}{\sqrt{\kappa+\eta}} & |E| > 2 \\ \sqrt{\kappa+\eta} & |E| \leqslant 2 \end{cases}. \tag{1.3}$$

Moreover, there is some constant $c > 0$ such that for $|z| \leqslant 10$ and $\mathrm{Im}[z] > 0$,

$$\begin{aligned} c \leqslant |m_{sc}(z)| &\leqslant 1 - c\eta \\ |1 - m_{sc}(z)^2| &\asymp \sqrt{\kappa+\eta}. \end{aligned} \tag{1.4}$$

Therefore, the above propositions mean that $|m_N(z) - X(Q)|, |Q - Y(Q)| \leqslant \frac{1}{N\sqrt{\kappa+\eta}}$. We first prove a bound on $Q - m_{sc}(z)$.

$$\begin{aligned} |Q - m_{sc}(z)| &\leqslant |Q - Y(Q)| + |Y(Q) - m_{sc}(z)| \\ &\approx \frac{N^{2\mathfrak{c}}}{N\sqrt{\kappa+\eta}} + Y'(m_{sc}(z))|Q - m_{sc}(z)|, \end{aligned}$$

where we use that the Taylor expansion converges sufficiently quickly.

We can reduce this by the above approximations that $Y'(m_{sc}) = m_{sc}^{2\ell+2}$. Therefore, we can approximate $|1 - m_{sc}^{2\ell+2}| \asymp \sqrt{\kappa+\eta}|1 + m_{sc}^2 + \cdots + m_{sc}^{2\ell}|$, and by toggling ℓ by 1, we can choose it such that this last term is at least constant. Therefore, we have

$$|Q - m_{sc}(z)| \lesssim |1 - Y'(m_{sc}(z))|^{-1} \cdot \frac{N^{2\mathfrak{c}}}{N\sqrt{\kappa+\eta}} \lesssim \frac{N^{2\mathfrak{c}}}{N(\kappa+\eta)}.$$

Given this, we can similarly expand

$$\begin{aligned} m_N(z) - m_d(z) &= m_N(z) - X(Q) + X(Q) - m_d(z) \\ &= \frac{1}{N\sqrt{\kappa+\eta}} + X'(m_{sc})(Q - m_{sc}) \\ &= O(\frac{N^{2\mathfrak{c}}}{N(\kappa+\eta)}). \end{aligned}$$

Recall our goal is to deduce that there is no eigenvalue at $2+N^{-2/3+4\mathfrak{c}}$. Therefore, we set $z = 2+\kappa+\mathrm{i}\eta$, where $\kappa = N^{-2/3+4\mathfrak{c}}$, and $\eta = N^{-2/3+\mathfrak{c}/2}$.

For our discrete operator, we always have that

$$\mathrm{Im}[m_N(z)] = \mathrm{Im}\left[\frac{1}{N}\sum_{i \in N} \frac{1}{\lambda_i - z}\right] = \frac{\eta}{N} \sum_{i \in [\![N]\!]} \frac{1}{|\lambda_i - E|^2 + \eta^2}.$$

If there is an eigenvalue at $2+\kappa$, then $\operatorname{Im}[m_N(z)] \geqslant 1/(N\eta) = N^{-1/3-\mathfrak{c}/2}$. On the other hand, by our above approximation, we have $\operatorname{Im}[m_d(z)] \asymp \eta/\sqrt{\kappa+\eta} \approx N^{-1/3-3\mathfrak{c}/2}$. In order to satisfy our given approximation, we must have $|m_N(z) - m_d(z)| \gtrsim N^{-1/3-\mathfrak{c}/2}$. However, our above calculation tells us that

$$|m_N(z) - m_d(z)| \lesssim N^{-1/3-2\mathfrak{c}},$$

meaning such an eigenvalue does not exist. To prove the theorem, we send $\mathfrak{c} \to 0$. □

Our goal in these notes is to show a weaker bound than this, however we will show this for *every* entry rather than just for the average given by the trace. Therefore, for $z \in \mathbb{C}^+$, we define the error parameter

Then our goal is as follows.

Theorem 1.16 ([2, Theorem 4.2])**.** *For any $0 < \mathfrak{c} < 0.25$, $r = (\mathfrak{c}/100)\log_{d-1}(N)$ and any $z \in \mathbb{C}^+$, we define $\eta(z) = \operatorname{Im}[z], \kappa(z) = \min\{|\operatorname{Re}[z]-2|, |\operatorname{Re}[z]+2|\}$.*

We define the error parameter

$$\varepsilon(z) := (\log N)^{100}\left(\frac{1}{(d-1)^r} + \sqrt{\frac{\operatorname{Im}[m_d(z)]}{N\eta(z)}} + \frac{1}{(N\eta(z))^{2/3}}\right). \tag{1.5}$$

With probability at least $1 - \mathrm{O}(N^{-1+o_N(1)})$, for every $i, j \in [\![N]\!]$, and any $z \in \mathbb{C}^+$ with $|z| \leqslant 1/\mathfrak{c}, \operatorname{Im}[z] \geqslant N^{-1+\mathfrak{c}}$,

$$\begin{aligned} &\left|G_{ij}(z) - G_{ij}\big(\operatorname{Ext}(\mathcal{B}_r(\{i,j\},\mathcal{G}), m_{\mathrm{sc}}(z)), z\big)\right| \leqslant \varepsilon(z), \\ &|Q(z) - m_{\mathrm{sc}}(z)|, \quad |m_N(z) - m_d(z)| \leqslant \frac{\varepsilon}{\sqrt{\kappa+\eta+\varepsilon}}. \end{aligned} \tag{1.6}$$

The term $\sqrt{\operatorname{Im} m_d(z)/(N\eta)}$ is the usual fluctuation scale for averaged Green's function quantities, while $(d-1)^{-r}$ measures the error from replacing the full graph by a radius-r neighborhood.

1.4 Improved bound [Extra reading]

We claim that this window $\operatorname{Im}[z]/N\eta$ is really the critical window. As we saw before, to see the effect of an individual eigenvalue, we need the difference of the two Stieltjes transforms to be $1/N\eta$. According to our above calculations, the difference of the Stietjes transforms given our error bound is $O(\frac{N^{2\mathfrak{c}}}{N(\kappa+\eta)})$. This error is fine when we work further than $N^{-2/3+o_N(1)}$ because of the stark difference in Green's functions, but when we move further in, we need a tighter bound to talk about individual eigenvalues. Nevertheless if we can get a tighter bound, then in fact, we can learn about the entire distribution, not just $N^{-2/3-\mathfrak{c}}$. However, to do this, we need to pass from a high probability bound to one on the distribution, as the distribution becomes nontrivial.

However, we cannot deduce anything tighter from examining just the Stieltjes transform $m_{sc}(z)$. This is because this represents the infinite limit, and there are no finite eigenvalues. Therefore, we cannot approximate to level $1/(N\eta)$ as desired. To make this approximate, we consider the GOE rather than the semicircle law corresponding to the infinite limit.

To find a self consistent equation for the finite GOE matrix versus its infinite counterpart the semicircle law, we first recall the equation for the semicircle law.

$$m_{sc}(z) = (-z - m_{sc})^{-1}.$$

Moving the righthand side over the left, this is equivalent to asking that

$$m_{sc}(z)^2 + zm + 1 = 0.$$

The spectrum of the finite GOE converges to that of the semicircle law in the limit. However, as it is a discrete distribution, for any fixed N it does not exactly satisfy the equation. Instead it has a corresponding correction term, which is the loop equation.

Proposition 1.17. *Define $s_N(z)$ to be the Stieltjes transform of $Z \sim GOE(N)$. Then it satisfies the following equations exactly.*

$$\mathbb{E}[s_N(z)^2 + z s_N(z) + 1 + \frac{\partial_z s_N(z)}{N}] = 0.$$

More generally, for any $p \geqslant 1$ and $z_1, \dots, z_{p-1} \in \mathbb{C}^+$,

$$\begin{aligned} &\mathbb{E}\left[\left(s_N(z)^2 + z s_N(z) + 1 + \frac{\partial_z s_N(z)}{N}\right)\prod_{i=1}^{p-1} s_N(z_j)\right] \\ &= -\frac{2}{N^2}\sum_{j=1}^{p-1}\mathbb{E}[\partial_{z_j}\frac{s_N(z) - s_N(z_j)}{z - z_j})\prod_{i\neq j} s_N(z_i)]. \end{aligned}$$

We claim that it will be sufficient to show a similar term for $m_N(z)$. Therefore, it will be sufficient to show a similar equation about our Q. We define our window to be

$$\mathbf{M} := \{w \in \mathbb{C} : N^{-2/3-\mathfrak{g}} \leqslant \mathrm{Im}[w] \leqslant N^{-2/3+\mathfrak{g}}, -N^{-2/3+\mathfrak{g}} \leqslant \mathrm{Re}[w] \leqslant N^{-2/3+\mathfrak{g}}\}. \tag{1.7}$$

We then define our region of interest as $\pm 2 + w$ for $w \in \mathbf{M}$.

Proposition 1.18. *There is an event Ω that occurs with probability at least $1 - N^{-1+o_N(1)}$ such that the following occurs. For any $z = \pm 2 + w$ for $w \in \mathbf{M}$,*

$$\begin{aligned} &\left|\mathbb{E}\left[\mathbf{1}(\mathcal{G} \in \Omega)\left(\frac{\mathcal{A}^2}{\ell+1}(Q(z) - Y_\ell(z, Q)) + \frac{\partial_z m_N(z)}{N}\right)\right]\right| \\ &\lesssim (d-1)^{-\ell/2} \cdot \frac{\mathrm{Im}[m_d(z)]}{N\eta}. \end{aligned} \tag{1.8}$$

The focus of the course will be to show this proposition. Note that to capture all the moments, we prove a more general statement.

Proposition 1.19. *More generally, take any $p \geqslant 1$, and $w, w_1, \dots w_{p-1} \in \mathbf{M}$. We then take $z = \pm 2 + w$ and $z_j = \pm 2 + w_j$ for each j.*

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}(\mathcal{G} \in \Omega)\left(\frac{\mathcal{A}^2}{\ell+1}(Q(z) - Y_\ell(z, Q)) + \frac{\partial_z m_N(z)}{N}\right)\left(\prod_{j=1}^{p-1}(m_N(z_j) - m_d(z_j))\right)\right] \\ &+ \sum_{j=1}^{p-1}\frac{2}{N^2}\mathbb{E}\left[\mathbf{1}(\mathcal{G} \in \Omega)\partial_{z_j}\left(\frac{m_N(z) - m_N(z_j)}{z - z_j}\right)\prod_{i\neq j}(m_N(z_i) - m_d(z_i))\right] \\ &= \mathrm{O}\left(N^{-(p-1)(1/3-\mathfrak{o})}(d-1)^{-\ell/2}\frac{\mathrm{Im}[m_d(z)]}{N\eta}\right). \end{aligned} \tag{1.9}$$

It is nontrivial to translate this into a distributional bound, but the novel technical achievement of the recent works is the above bounds.

2 Resampling procedure

The core of much of probability theory is to split analysis into a series of independent events. The tricky thing about the random regular graph model is that the independent events are not the entries of the matrix. Indeed, the knowledge that any given edge exists in the graph greatly affects the probability of existence of other edges elsewhere.

The source of randomness we exploit is not independence of matrix entries, but rather the invariance of the uniform random regular graph under admissible edge switchings. Specifically, if the graph contains the disjoint edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$, and if the proposed new edges do not already exist, then replacing them by $\{v_1, v_4\}$ and $\{v_2, v_3\}$ gives another simple d-regular graph. Since the uniform measure is invariant under this involution, the original graph and the switched graph are equally likely.

We will use this by performing the edge switch on an entire neighborhood all at once.

2.1 Resampling

For any graph $\mathcal{G}$, we denote the set of unoriented edges by $E(\mathcal{G})$, and the set of oriented edges by $\vec{E}(\mathcal{G}) := \{(u,v),(v,u) : \{u,v\} \in E(\mathcal{G})\}$. For a subset $\vec{S} \subset \vec{E}(\mathcal{G})$, we denote by S the set of corresponding non-oriented edges. For a subset $S \subset E(\mathcal{G})$ of edges we denote by $[S] \subset [\![N]\!]$ the set of vertices incident to any edge in S. Moreover, for a subset $\mathbb{V} \subset [\![N]\!]$ of vertices, we define $E(\mathcal{G})|_{\mathbb{V}}$ to be the set of edges of $\mathcal{G}$ with both endpoints in $\mathbb{V}$.

Definition 2.1. *A (simple) switching is encoded by two oriented edges $\vec{S} = \{(v_1,v_2),(v_3,v_4)\} \subset \vec{E}$. We assume that the two edges are disjoint, i.e. that $|\{v_1,v_2,v_3,v_4\}| = 4$. Then the switching consists of replacing the edges $\{v_1,v_2\},\{v_3,v_4\}$ with the edges $\{v_1,v_4\},\{v_2,v_3\}$. We denote the graph after the switching $\vec{S}$ by $T_{\vec{S}}(\mathcal{G})$, and the new edges $\vec{S}' = \{(v_1,v_4),(v_2,v_3)\}$ by $T(\vec{S}) = \vec{S}'$.*

The key to the idea of the local resampling is that the neighborhoods of most vertices are trees. Therefore, we can choose swappings in such a way that all the local neighborhoods are the same, the only part that changes is how they are connected to each other.

The local resampling involves a fixed center vertex, which we now assume to be vertex o, and a radius ℓ. Given a d-regular graph $\mathcal{G}$, we write $\mathcal{T} := \mathcal{B}_\ell(o,\mathcal{G})$ to denote the radius-ℓ neighborhood of o (which may not necessarily be a tree) and write $\mathbb{T}$ for its vertex set. The edge boundary $\partial_E\mathcal{T}$ of $\mathcal{T}$ consists of the edges in $\mathcal{G}$ with one vertex in $\mathbb{T}$ and the other vertex in $[\![N]\!] \setminus \mathbb{T}$. We enumerate the edges of $\partial_E\mathcal{T}$ as $\partial_E\mathcal{T} = \{e_1,e_2,\dots,e_\mu\}$, where $e_\alpha = \{l_\alpha,a_\alpha\}$ with $l_\alpha \in \mathbb{T}$ and $a_\alpha \in [\![N]\!] \setminus \mathbb{T}$. We orient the edges e_α by defining $\vec{e}_\alpha = (l_\alpha,a_\alpha)$. We notice that μ and the edges $e_1,e_2,\dots,e_\mu$ depend on $\mathcal{G}$. The edges e_α are distinct, but the vertices a_α are not necessarily distinct and neither are the vertices l_α. Our local resampling switches the edge boundary of $\mathcal{T}$ with randomly chosen edges in $\mathcal{G}^{(\mathbb{T})}$ if the switching is admissible (see below), and leaves them in place otherwise. We choose $(b_1,c_1),\dots,(b_\mu,c_\mu)$ to be independent, uniformly chosen oriented edges from the graph $\mathcal{G}^{(\mathbb{T})}$, i.e., the oriented edges of $\mathcal{G}$ that are not incident to $\mathbb{T}$, and define

$$\vec{S}_\alpha = \{\vec{e}_\alpha,(b_\alpha,c_\alpha)\}, \qquad \mathbf{S} = (\vec{S}_1,\vec{S}_2,\dots,\vec{S}_\mu). \tag{2.1}$$

The sets $\mathbf{S}$ will be called the *resampling data* for $\mathcal{G}$. We remark that repetitions are allowed in the data $(b_1,c_1),(b_2,c_2),\cdots,(b_\mu,c_\mu)$. We define an indicator that will be crucial to the definition of the switch.

Definition 2.2. *For $\alpha \in [\![\mu]\!]$, we define the indicator functions $I_\alpha \equiv I_\alpha(\mathcal{G},\mathbf{S}) = 1$ if*

1. *the subgraph $\mathcal{B}_{\mathfrak{R}/4}(\{a_\alpha,b_\alpha,c_\alpha\},\mathcal{G}^{(\mathbb{T})})$ after adding the edge $\{a_\alpha,b_\alpha\}$ is a tree;*
2. *and $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(\{a_\alpha,b_\alpha,c_\alpha\},\{a_\beta,b_\beta,c_\beta\}) > \mathfrak{R}/4$ for all $\beta \in [\![\mu]\!] \setminus \{\alpha\}$.*

The indicator function I_α imposes two conditions. The first one is a "tree" condition, which ensures that a_α and $\{b_\alpha,c_\alpha\}$ are far away from each other, and their neighborhoods are trees. The second one

imposes an "isolation" condition, which ensures that we only perform simple switching when the switching pair is far away from other switching pairs. In this way, we do not need to keep track of the interaction between different simple switchings.

We define the *admissible set*

$$\mathsf{W}_{\mathbf{S}} := \{\alpha \in [\![\mu]\!] : I_\alpha(\mathcal{G}, \mathbf{S})\}. \tag{2.2}$$

We say that the index $\alpha \in [\![\mu]\!]$ is *switchable* if $\alpha \in \mathsf{W}_{\mathbf{S}}$. We denote the set $\mathbb{W}_{\mathbf{S}} = \{b_\alpha : \alpha \in \mathsf{W}_{\mathbf{S}}\}$. Let $\nu := |\mathsf{W}_{\mathbf{S}}|$ be the number of admissible switchings and $\alpha_1, \alpha_2, \dots, \alpha_\nu$ be an arbitrary enumeration of $\mathsf{W}_{\mathbf{S}}$. Then we define the switched graph by

$$T_{\mathbf{S}}(\mathcal{G}) := \left(T_{\vec{S}_{\alpha_1}} \circ \cdots \circ T_{\vec{S}_{\alpha_\nu}}\right)(\mathcal{G}), \tag{2.3}$$

and the resampling data by

$$T(\mathbf{S}) := (T_1(\vec{S}_1), \dots, T_\mu(\vec{S}_\mu)), \quad T_\alpha(\vec{S}_\alpha) := \begin{cases} T(\vec{S}_\alpha) & (\alpha \in \mathsf{W}_{\mathbf{S}}), \\ \vec{S}_\alpha & (\alpha \notin \mathsf{W}_{\mathbf{S}}). \end{cases} \tag{2.4}$$

Because the admissible switching pairs are separated from one another, the corresponding switchings do not interact; in particular, the final graph is independent of the chosen enumeration of $\mathsf{W}_{\mathbf{S}}$.

In the following lemma we show that typically W is very large.

Lemma 2.3. *Assuming $\mathcal{G} \in \overline{\Omega}$, with probability $1 - N^{-\omega_N(1)}$, $\mu - |\mathsf{W}| \leqslant \log N$. Moreover, if o has a tree neighborhood of radius $\mathfrak{R}$, and $\ell \leqslant \mathfrak{R}$, then with probability $1 - O(N^{-1+2\mathfrak{c}})$, we have $|\mathsf{W}| = \mu$.*

Proof. We only prove the first statement, as the second is analogous. By the assumption that $\mathcal{G}$ is $\mathfrak{R}$-tangle free, the $\mathfrak{R}$-neighborhood of o has at most one cycle. Moreover, there cannot be a cycle that is contained in $\mathcal{B}_{\mathfrak{R}/4}(a_\alpha, \mathcal{G}^{(\mathbb{T})})$ for two different values of α, as that will create two cycles in the $\mathfrak{R}$-neighborhood of o, one in $\mathcal{G}^{(\mathbb{T})}$ and another in $\mathcal{G}$ containing o. Thus there is at most one a_α such that $\mathcal{B}_{\mathfrak{R}/4}(a_\alpha, \mathcal{G}^{(\mathbb{T})})$ is not a tree.

Suppose a_α has a tree neighborhood of radius $\mathfrak{R}/4$. As we will see, there are at most $N^{\mathfrak{c}}$ choices of b_α that do not have tree neighborhood of radius $\mathfrak{R}$. Assuming we have not made such a choice, all conditions of I_α are satisfied if $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_\alpha, a_\alpha \cup \{b_\beta\}_{\beta \neq \alpha}) \geqslant \mathfrak{R}/4 + 1$. The number of vertices v such that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(v, a_\alpha \cup \{b_\beta\}_{\beta \neq \alpha}) \leqslant \mathfrak{R}/4$ is at most $d(d-1)^\ell d(d-1)^{\frac{\mathfrak{c}}{4}\log N} \leqslant N^{\mathfrak{c}/2}$. As the choice of each b_α is independent, the probability that we choose at least $\log N - 1$ many bad b_α is at most $\binom{\mu}{\log N - 1} N^{-\log N(1-\mathfrak{c})} \leqslant (d-1)^{\log N \ell} N^{-\log N(1-\mathfrak{c})} = N^{-\omega_N(1)}$. □

Definition 2.4. *Let*

$$\widetilde{\mathcal{G}} := T_{\mathbf{S}}(\mathcal{G}).$$

For any graph-dependent quantity $F = F(\mathcal{G}, z)$, we write

$$\widetilde{F} := F(\widetilde{\mathcal{G}}, z).$$

In particular,

$$\widetilde{G}(z) := G(z, \widetilde{\mathcal{G}}), \qquad \widetilde{Q}(z) := Q(\widetilde{\mathcal{G}}, z).$$

Proposition 2.5. *The joint distribution satisfies*

$$(\mathcal{G}, T_{\mathbf{S}}(\mathcal{G})) \overset{law}{=} (T_{\mathbf{S}}(\mathcal{G}), \mathcal{G}).$$

Proof. The map

$$(\mathcal{G}, \mathbf{S}) \mapsto (T_{\mathbf{S}}(\mathcal{G}), T(\mathbf{S}))$$

is an involution on the enlarged probability space. Indeed, after performing the admissible switchings, the transformed data $T(\mathbf{S})$ records precisely the reverse switchings. The admissibility conditions ensure that the neighborhood $\mathcal{T}$ is unchanged by the switching, so the transformed data $T(\mathbf{S})$ is admissible resampling data for the switched graph. Hence,

$$T_{T(\mathbf{S})}(T_{\mathbf{S}}(\mathcal{G})) = \mathcal{G}.$$

Moreover, the admissibility conditions are symmetric under this transformation, and the uniform measure on d-regular graphs together with the uniform choice of the resampling data is preserved. Therefore the pair

$$(\mathcal{G}, T_{\mathbf{S}}(\mathcal{G}))$$

has the same distribution as

$$(T_{\mathbf{S}}(\mathcal{G}), \mathcal{G}).$$

□

2.2 Resolvent Expansion After Switching

Applying the Schur complement formula with respect to the decomposition $\mathbb{T} \sqcup \mathbb{T}^c$, we obtain

$$\begin{aligned} & G_{oo} \\ &= (H|_{\mathbb{T}} - z\mathbb{I} - H|_{\mathbb{T}\mathbb{T}^c} G^{(\mathbb{T})} H|_{\mathbb{T}^c\mathbb{T}})^{-1}_{oo} \\ &= (H|_{\mathbb{T}} - z\mathbb{I} - \frac{1}{d-1} \sum_{\alpha,\beta} G^{(\mathbb{T})}_{a_\alpha a_\beta} e_{l_\alpha} e^*_{l_\beta})^{-1}_{oo} \end{aligned}$$

Now assume that the local neighborhood of the graph is tree-like, which is true for most vertices. Then a_α is close to o only through l_α, so after deleting vertices, the off-diagonal terms are far away. Therefore, as we will show, we can make the approximation

$$\begin{aligned} & G_{oo} \\ &\approx (H|_{\mathbb{T}} - z\mathbb{I} - \frac{1}{d-1} \sum_{\alpha} G^{(l_\alpha)}_{a_\alpha a_\alpha} e_{l_\alpha} e^*_{l_\alpha})^{-1}_{oo}. \end{aligned}$$

Therefore, up to some error term we can see G_{oo} as a function of the directed edges (l_α, a_α) in the graph. We want to change this to a randomly selected set of edges. Specifically, we would like to instead take

$$(H|_{\mathbb{T}} - z\mathbb{I} - \frac{1}{d-1} \sum_{\alpha} G^{(b_\alpha)}_{c_\alpha c_\alpha} e_{l_\alpha} e^*_{l_\alpha})^{-1}_{oo},$$

for randomly selected vertices in the graph. However, we want to do so in a way that respects the distribution of the graph. What we will show is that, in fact

$$\widetilde{G}_{oo} \approx (H|_{\mathbb{T}} - z\mathbb{I} - \frac{1}{d-1} \sum_{\alpha} \widetilde{G}^{(b_\alpha)}_{c_\alpha c_\alpha} e_{l_\alpha} e^*_{l_\alpha})^{-1}_{oo}.$$

We have succeeded in minimally changing the graph through the switch, but there are still differences, and we need to relate the Green's functions of the two graphs.

Our goal will be to write the previously introduced Schur complement. The key to using this formula is to track the rank of the perturbation to our operator H after performing the switch. To do this, we realize that we can write the change in our operator as

$$\widetilde{H} - H = \sum_{\alpha \in [\mu]} \xi_\alpha, \qquad \xi_\alpha := \frac{1}{\sqrt{d-1}} (-\Delta_{l_\alpha a_\alpha} - \Delta_{b_\alpha c_\alpha} + \Delta_{l_\alpha c_\alpha} + \Delta_{a_\alpha b_\alpha}).$$

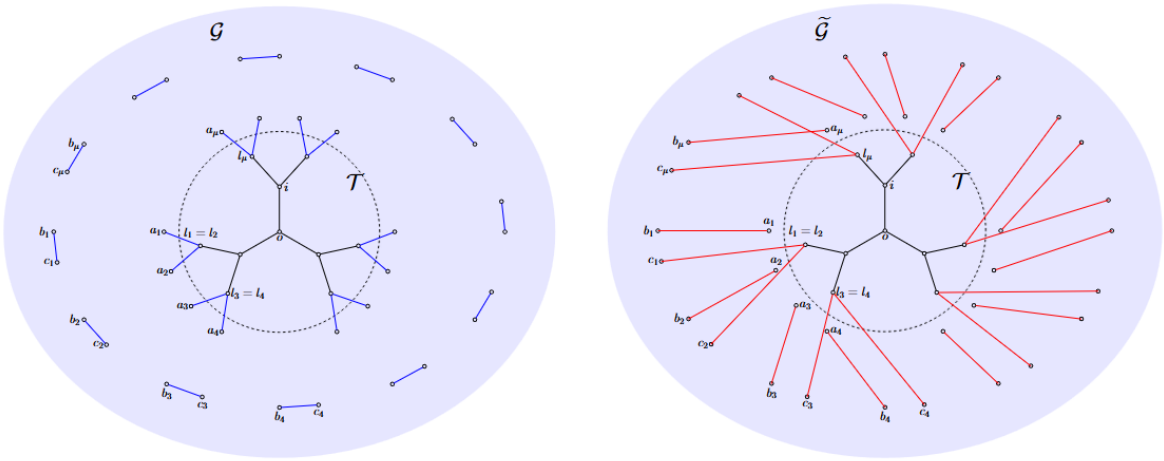


Figure 3: An example of the switching mechanism

Therefore this is a low rank update. Moreover, the switch is only performed on the vertex set $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_\alpha$. Looking a little more carefully, we see that every edge changed by the switch is incident to one of the vertices in $\{l_\alpha, b_\alpha\}_\alpha$. Therefore, as $\mathbb{W} = \{b_\alpha\}_\alpha$, and $\{l_\alpha\} \subseteq \mathbb{T}$, this means that $H^{(\mathbb{TW})} = \widetilde{H}^{(\mathbb{TW})}$, and therefore, $G^{(\mathbb{TW})} = \widetilde{G}^{(\mathbb{TW})}$.

Therefore, if we fix the central neighborhood, we can write this as a function of the boundary vertices.

We use the Schur complement formula again to write

$$\widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\beta} = \widetilde{G}^{(\mathbb{TW})}_{c_\alpha c_\beta} + (\widetilde{G}^{(\mathbb{T})}(\widetilde{G}^{(\mathbb{T})}|_{\mathbb{W}})^{-1}\widetilde{G}^{(\mathbb{T})})_{c_\alpha c_\beta}.$$

The main term is the first term. However, we realize that locally around c_α and c_β, the local behavior is that their neighbors b_α and b_β have been deleted, whereas everything else is far away. Therefore we write

$$\widetilde{G}^{(\mathbb{TW})}_{c_\alpha c_\beta} = G^{(\mathbb{TW})}_{c_\alpha c_\beta} = G^{(b_\alpha b_\beta)}_{c_\alpha c_\beta} - (G^{(b_\alpha b_\beta)}(G^{(b_\alpha b_\beta)}|_{\mathbb{TW}\setminus\{b_\alpha, b_\beta\}})^{-1}G^{(b_\alpha b_\beta)})_{c_\alpha c_\beta}.$$

by the Schur complement formula.

This is the basic reduction. The remaining work is to show that the error terms produced by these Schur complement expansions are small, and that the main terms can be averaged over the additional randomness introduced by the resampling.

One important tool in doing this is the resolvent identity, allowing us to compare matrix inverses. Specifically,

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} = B^{-1}(B - A)A^{-1}.$$

A simple application is the Ward identity.

Lemma 2.6 (The Ward Identity)**.** *It is the case that*

$$\sum_{y \in [\![N]\!]} |G_{xy}|^2 = \mathrm{Im}[G_{xx}]/\eta. \tag{2.5}$$

Proof. Since $G = (H - z)^{-1}$ and $G^* = (H - \overline{z})^{-1}$, the resolvent identity gives

$$G - G^* = G\big((H - \overline{z}) - (H - z)\big)G^* = 2\mathrm{i}\eta\, GG^*.$$

Therefore

$$\operatorname{Im} G = \eta G G^*.$$

Taking the xx-entry gives

$$\operatorname{Im} G_{xx} = \eta \sum_{y \in [\![N]\!]} |G_{xy}|^2,$$

which proves the claim. □

3 Concentration of the Green's function

What we'll do today is how to show when given a certain bound on the Green's function, we can improve it. This is a bootstrapping argument.

Specifically, we fix some well chosen error term.

Definition 3.1 (Choices of parameters)**.** *We fix* $\mathfrak{c} := 0.02$. *Let* $\mathfrak{R} = (\mathfrak{c}/4)\log_{d-1} N$, $r = (\mathfrak{c}/32)\log_{d-1} N$ *and* $\ell \in [12\log_{d-1}\log N, 24\log_{d-1}\log N]$. *With this choice,*

$$\mathfrak{R}/8 = r = O(\log N) \gg \ell = \Theta(\log\log N), \quad (\log N)^{12} \leqslant (d-1)^{\ell} \leqslant (\log N)^{24}. \tag{3.1}$$

We set our error term to be

$$\varepsilon(z) := (\log N)^{100}\left(\frac{1}{(d-1)^r} + \sqrt{\frac{\operatorname{Im}[m_d(z)]}{N\eta(z)}} + \frac{1}{(N\eta(z))^{2/3}}\right).$$

We create a smaller error term

$$\varphi(z) := (\log N)^{24}\sqrt{\frac{\operatorname{Im}[m_d(z)] + \varepsilon(z)/\sqrt{\kappa(z)+\eta(z)+\varepsilon(z)}}{N\eta(z)}}. \tag{3.2}$$

Notice that from our choice of parameters, on the spectral domain $z \in \mathbb{C}^+$ with $\operatorname{Im}[z] \geqslant (\log N)^{300}/N$, we have

$$\varepsilon(z) \leqslant \frac{1}{(\log N)^{48}}, \quad \varphi(z) \leqslant \frac{\varepsilon(z)}{(\log N)^{48}}. \tag{3.3}$$

We will assume that our functions satisfy a bound given ϵ,

Definition 3.2. *Fix* z *such that* $\operatorname{Im}[z] \geqslant N^{-1+\mathfrak{c}}$. *The event* $\Omega(z) \subset \overline{\Omega}$ *is such that for every* $i, j \in [N]$,

$$\begin{aligned} &\left|G_{ij}(z) - G_{ij}\big(\operatorname{Ext}(\mathcal{B}_r(\{i,j\},\mathcal{G}), m_{\mathrm{sc}}(z)), z\big)\right| \leqslant \varepsilon(z), \\ &|Q(z) - m_{\mathrm{sc}}(z)|, \quad |m_N(z) - m_d(z)| \leqslant \frac{\varepsilon}{\sqrt{\kappa+\eta+\varepsilon}}. \end{aligned} \tag{3.4}$$

To show Theorem 1.16, we will bootstrap by showing that being in Ω implies that with high probability, we can improve our error. This will allow us to reason about the statement for different z by using that entries of the Green's function are $\frac{1}{\eta}$-Lipschitz.

Definition 3.3. *Define* $\Omega_o^-(\mathcal{G}, z) \subseteq \Omega(z)$ *to be the event that the randomly selected graph* $\mathcal{G}$ *satisfies the stronger bounds*

$$|Q(\mathcal{G}, z) - m_{sc}(z)| \leqslant \frac{\varepsilon(z)}{\log N\sqrt{\kappa+\eta+\varepsilon}}, \tag{3.5}$$

$$|G_{oi}(\mathcal{G}, z) - G_{oi}(\operatorname{Ext}(\mathcal{B}_r(\{o,i\},\mathcal{G}), Q(\mathcal{G}, z)), z)| \leqslant \frac{\varepsilon}{(\log N)^2} \qquad \text{for all } i \in [N]. \tag{3.6}$$

The bootstrapping follows by the following proposition.

Proposition 3.4. *Assume that $\mathcal{G} \in \Omega$. We then perform the local resampling at depth ℓ around vertex o. For any fixed $\mathfrak{q} > 0$, with probability $1 - O(N^{-\mathfrak{q}})$, we have that the resampled graph $\widetilde{\mathcal{G}} \in \Omega_o^-(z)$.*

As the distribution of the random regular graph is invariant under edge swaps, taking a union bound over all vertices immediately implies the following.

Corollary 3.5. *Under the assumptions of Theorem 3.4, with probability $1 - O(N^{-\mathfrak{q}+1})$,*

$$|Q(\mathcal{G}, z) - m_{sc}(z)| \leqslant \frac{\varepsilon(z)}{\log N \sqrt{\kappa + \eta + \varepsilon}},$$
$$|G_{ij}(\mathcal{G}, z) - G_{ij}(\mathrm{Ext}(\mathcal{B}_r(\{i,j\}, \mathcal{G}), Q(\mathcal{G}, z)), z)| \leqslant \frac{\varepsilon}{(\log N)^2} \qquad \text{for all } i, j \in [N].$$

For an idea of how to use this to prove our entrywise bound, we start with $|z| \geqslant 2d$ for which the entrywise bound is true through the Combes–Thomas method. We then bootstrap by applying Theorem 3.5. Entries of the Green's function are Lipschitz with constant $1/\operatorname{Im}[z]$. Therefore, we know that $\Omega(z')$ holds for any $|z' - z| \leqslant \operatorname{Im}[z]/2$. We then can apply Theorem 3.5 for $z' = E + \mathrm{i}\eta/2$. We repeat this until we reach $\eta \approx \log^{300} N/N$. We apply this process to create a net of z where Theorem 3.5 is satisfied, giving $\Omega(z)$ for the entire region.

Our goal is write $\widetilde{G}_{oo}$ as a series of independent terms that come from the switching, and exist in the random graph.

The first point is to assume that everything is bounded by ε, we will show the new graph has coefficients bounded by $\epsilon/\log N$. In order to do this, we use the series of transformations

$$G \to G^{(\mathbb{T})} \to G^{(\mathbb{TW})} = \widetilde{G}^{(\mathbb{TW})} \to \widetilde{G}^{(\mathbb{T})} \to \widetilde{G}.$$

We claim that in the first two steps, the entrywise bound on G translates into an entrywise bound on $G^{(\mathbb{V})}$. Then, the randomness over the choice of $\widetilde{G}$ means that we get an improvement in the latter steps.

The basic tools we will use are as follows.

1. Distance bound: randomly selected vertices are typically far from each other in graph distance. This means that local neighborhoods of (b_α, c_α) can, up to small error, be treated independently.
2. Green's function bound: by the Ward identity (2.5), most pairs of vertices have Green's function entry at most φ, for φ as defined in (3.2).

By the Ward identity, we have

$$\frac{1}{N}\sum_{j \in V} |G_{ij}|^2 = \frac{\operatorname{Im}[G_{ii}]}{N\eta} \lesssim \frac{\operatorname{Im}[m_d(z)] + \varepsilon/\sqrt{\kappa + \eta + \varepsilon}}{N\eta}.$$

Therefore, there are at most $N/\log^{48} N$ vertices j such that $|G_{ij}| \geqslant \varphi$. This means that under the choice of the switched vertices, typically no choices will have entry greater than φ, and we can use this to signify typical Green's function behavior.

Proof Sketch of Theorem 3.4. We will give the simplest case here. Other cases generate other error terms, but our goal is to give the general argument. This is the case showing that $\widetilde{G}_{oo}$ approximates $X_\ell(Q)$, when o has a tree neighborhood of depth ℓ.

The first idea is that if vertices are not obviously close to each other, then the corresponding Green's function term is negligible. This gets rid of all off-diagonal terms.

$$\widetilde{G}_{oo} = (-z + H|_{\mathbb{T}} - \frac{1}{d-1}\sum_{\alpha,\beta} \widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\beta} e_{l_\alpha} e^*_{l_\beta})^{-1}$$
$$\approx (-z + H|_{\mathbb{T}} - \frac{1}{d-1}\sum_{\alpha} \widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\alpha} e_{l_\alpha} e^*_{l_\alpha})^{-1}$$

The next idea is that locally, from the perspective of c_α, we have deleted its neighbor l_α. This is the same behavior in the original graph, if we have deleted its neighbor b_α. Therefore, we can take

$$(-z + H|_{\mathbb{T}} - \frac{1}{d-1}\sum_{\alpha} \widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\alpha} e_{l_\alpha} e^*_{l_\alpha})^{-1} \approx (-z + H|_{\mathbb{T}} - \frac{1}{d-1}\sum_{\alpha} G^{(\mathbb{T}b_\alpha)}_{c_\alpha c_\alpha} e_{l_\alpha} e^*_{l_\alpha})^{-1}.$$

We then want to compare this versus the extension. We define

$$\widetilde{P}_{oo} = G(\mathrm{Ext}(B_\ell(o, \mathcal{G}), \widetilde{Q}), z) = (-z + H|_{\mathbb{T}} - \frac{\widetilde{Q}}{d-1}\sum_{\alpha} e_{l_\alpha} e^*_{l_\alpha})^{-1}.$$

Note $\widetilde{P}_{oo} = X_\ell(\widetilde{Q})$.

Assuming that $\widetilde{Q} \approx Q$,

$$\begin{aligned}
(\widetilde{G} - \widetilde{P})_{oo} &= \frac{1}{d-1}\sum_{\alpha\beta} \widetilde{G}_{ol_\alpha}(\widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\beta} - \widetilde{P}^{(\mathbb{T})}_{c_\alpha c_\beta})\widetilde{P}_{l_\beta o} \\
&= \frac{1}{d-1}\sum_{\alpha} \widetilde{P}_{ol_\alpha}(\widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\alpha} - \widetilde{Q})\widetilde{P}_{l_\alpha o} \\
&\approx \frac{1}{d-1}\sum_{\alpha} \widetilde{P}_{ol_\alpha}(G^{(\mathbb{T}b_\alpha)}_{c_\alpha c_\alpha} - Q)\widetilde{P}_{l_\alpha o} \\
&\approx \frac{1}{d-1}\sum_{\alpha} P^2_{ol_\alpha}(G^{(\mathbb{T}b_\alpha)}_{c_\alpha c_\alpha} - Q).
\end{aligned}$$

This then becomes like a weighted average. The weights are $P^2_{ol_\alpha}$. In the exercise session, we show that if $Q = m_{sc}(z)$, then these weights are of order $(d-1)^{-\ell}$. Using a Taylor expansion, we can show that this is still the case for Q close to $m_{sc}(z)$.

Therefore, we have a series of independent random variables, we just need to show that their expectation is 0. Up to the negligible error coming from restricting the sampled edge to $\mathcal{G}^{(\mathbb{T})}$, the expectation of $G^{(b_\alpha)}_{c_\alpha c_\alpha}$ over a uniformly chosen oriented edge (b_α, c_α) is Q. Indeed, reversing a uniformly chosen oriented edge is still uniform, and Q averages $G^{(u)}_{vv}$ over directed edges (v, u). We have

$$G^{(\mathbb{T}b_\alpha)}_{c_\alpha c_\alpha} = G^{(b_\alpha)}_{c_\alpha c_\alpha} - \sum_{x,y\in\mathbb{T}} G^{(b_\alpha)}_{c_\alpha x}(G^{(b_\alpha)}|_{\mathbb{T}})^{-1}_{xy} G^{(b_\alpha)}_{y c_\alpha}.$$

We claim that this last error term is negligible. Given that, by a standard tail bound on an independent sum (say an Azuma bound) we recover that

$$|(\widetilde{G} - \widetilde{P})_{oo}| \lesssim \log N (d-1)^{-\ell/2}\epsilon.$$

□

Without the assumption that the neighborhood is tree-like, the weights are more complicated, as we have to pass from the extension at one level to the extension at another. Nevertheless, a similar method works.

Our main tool for performing the above calculations was the resolvent identity and the Ward identity. I'll focus on two approximations that we made in the above argument sketch.

Lemma 3.6 ([2, Proposition 5.22]). *Assume $\mathcal{G} \in \Omega(z)$. With probability $1 - O(N^{-\mathfrak{q}})$ over the choice of the switch, we have*

$$|Q(\mathcal{G}, z) - Q(\widetilde{\mathcal{G}}, z)| \lesssim \frac{(d-1)^{2\ell}\left(\mathrm{Im}[m_d] + \varepsilon + \varepsilon/\sqrt{\kappa+\eta+\varepsilon}\right)}{N\eta} + N^{-1+\mathfrak{c}}. \tag{3.7}$$

Proof. We can first reduce to the set of vertices j that are away from vertices involved in the sampling. Specifically, we say $\chi_j(\mathcal{G}^{(\mathbb{TW})})$ is the indicator that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{TW})}}(j, \{a_\alpha, b_\alpha, c_\alpha, l_\alpha\}_{\alpha\in[\![\mu]\!]}) > \mathfrak{R}/4$. Then we can write

$$\begin{aligned}
&\frac{1}{Nd}\sum_{i\sim j} \widetilde{G}^{(j)}_{ii} - G^{(j)}_{ii} \\
&= \frac{1}{Nd}\sum_{i\sim j} \chi_j(\mathcal{G}^{(\mathbb{TW})})(\widetilde{G}^{(j)}_{ii} - G^{(j)}_{ii}) + O(N^{-1+\mathfrak{c}}) \\
&= \frac{1}{Nd}\sum_{i\sim j}\sum_{x,y\in\mathbb{TW}} \chi_j(\mathcal{G}^{(\mathbb{TW})})(\widetilde{G}^{(j)}_{ix}(\widetilde{G}^{(j)}|_{\mathbb{TW}})^{-1}_{xy}\widetilde{G}^{(j)}_{yi} - G^{(j)}_{ix}(G^{(j)}|_{\mathbb{TW}})^{-1}_{xy}G^{(j)}_{yi}) + O(N^{-1+\mathfrak{c}}).
\end{aligned}$$

The first step is to show that $|(G^{(j)}|_{\mathbb{TW}})^{-1}_{xy}|$ is not too large. Thus we define $P := G(\mathrm{Ext}(\mathcal{B}_r(\mathbb{TW}, \mathcal{G}), m_{sc}), z)$ and expand

$$(G^{(j)}|_{\mathbb{TW}})^{-1} = \sum_{k=0}^{\infty}(P|_{\mathbb{TW}})^{-1}\left((P|_{\mathbb{TW}} - G^{(j)}|_{\mathbb{TW}})(P|_{\mathbb{TW}})^{-1}\right)^k. \tag{3.8}$$

To bound this, we can approximate P

$$\|(P|_{\mathbb{T}})^{-1}\| = \|H|_{\mathbb{T}} - z - B^* m_{sc}(z) B\| \lesssim 1.$$

Therefore, splitting $(G^{(v)}|_{\mathbb{T}})^{-1}$ in (3.8) according to $k = 0$ and $k > 0$ gives that for $x, y \in \mathbb{T}$, we have

$$|(G^{(v)}|_{\mathbb{T}})^{-1}_{xy}| = |(P|_{\mathbb{T}})^{-1}_{xy}| + O(\varepsilon) \lesssim 1 \tag{3.9}$$

where we used that $(P|_{\mathbb{T}} - G^{(v)}|_{\mathbb{T}})_{xy} = O(\varepsilon)$ by Ω. Since $|\mathbb{TW}| \lesssim (d-1)^\ell$ and $(d-1)^\ell\varepsilon \ll 1$, the entrywise error gives an operator-norm error small enough for the Neumann series to converge.

Finally, we can write the rest similar to before as

$$\begin{aligned}
\sum_{x,y\in\mathbb{TW}}\sum_{i\sim j} G_{xi}G_{yj} &\lesssim \frac{1}{2}\sum_{x,y\in\mathbb{TW}}\sum_{i\sim j} |G_{xi}|^2 + |G_{yj}|^2 \\
&\lesssim (d-1)^\ell \sum_{x\in\mathbb{TW}} \frac{\mathrm{Im}[G_{xx}]}{N\eta} \\
&\lesssim (d-1)^{2\ell}\frac{\mathrm{Im}[m_d] + \varepsilon + \varepsilon/\sqrt{\kappa+\eta+\varepsilon}}{N\eta}
\end{aligned}$$

by our assumptions on the entrywise bound.

□

The other is the bound on $\widetilde{G}_{c_\alpha c_\alpha}$.

Definition 3.7. *The set $\mathsf{U} \subseteq \mathsf{W}$ is the set of indices α such that there is no $\beta \in \mathsf{W}$, $\beta \neq \alpha$ where*

$$\mathrm{dist}_{\mathcal{G}}(b_\alpha, b_\beta) \leqslant \mathfrak{R}/4 \tag{3.10}$$

or

$$|G_{b_\alpha b_\beta}| \geqslant \varphi, \tag{3.11}$$

where $\mathfrak{R}$ and φ are defined in Theorem 3.1.

Proposition 3.8 ([2, Proposition 5.10, Proposition 5.18 (5.51-5.53) and Proposition 6.3]). *Let $\mathcal{G} \in \Omega$. For any fixed $\mathfrak{q} > 0$, with probability at least $1 - O(N^{-\mathfrak{q}})$ over the choice of resampling data $\mathbf{S}$, the following holds.*

1. $|\mathsf{W} \setminus \mathsf{U}| \lesssim \log N$, *with* U *defined in Theorem 3.7.*
2. *for any fixed* $\alpha \in \mathsf{W} \setminus \mathsf{U}$, *at most* $O_{\mathfrak{q}}(1)$ *indices* $\beta \in \mathsf{W}$ *satisfy* (3.10) *or* (3.11).
3. *We have for* $\alpha \in \mathsf{U}$

$$|\widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\beta}(z)| \lesssim (\varepsilon(z))^2 + \varphi(z), \qquad \text{if } \beta \in [1, \mu] \setminus \mathsf{U}, \tag{3.12}$$

$$|\widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\beta}(z)| \lesssim (\varepsilon(z))^3 + \varphi(z), \qquad \text{if } \beta \in \mathsf{U} \setminus \{\alpha\}, \tag{3.13}$$

$$|\widetilde{G}^{(\mathbb{T})}_{c_\alpha i}(z)| \lesssim (\varepsilon(z))^2 + \varphi(z), \qquad \text{if } |G_{ic_\alpha}|, |G_{ib_\alpha}| \leqslant \varphi \text{ and } \operatorname{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, a_\alpha, b_\alpha, c_\alpha) \geqslant \mathfrak{R}/4. \tag{3.14}$$

4. *For any index* $\alpha \in \mathsf{U}$, *we have*

$$|\widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\alpha} - G^{(\mathbb{T}b_\alpha)}_{c_\alpha c_\alpha}| \lesssim (d-1)^\ell \varphi^2. \tag{3.15}$$

Proof of 4. We can write

$$\begin{aligned}
&\widetilde{G}^{(\mathbb{T})}_{c_\alpha c_\alpha} - G^{(\mathbb{T}b_\alpha)}_{c_\alpha c_\alpha} \\
&= \widetilde{G}^{(\mathbb{T}\mathbb{W})}_{c_\alpha c_\alpha} - G^{(\mathbb{T}\mathbb{W})}_{c_\alpha c_\alpha} \\
&\quad - \sum_{x,y \in \mathbb{W}\setminus\{b_\alpha\}} \left(\widetilde{G}^{(\mathbb{T})}_{c_\alpha x} (\widetilde{G}^{(\mathbb{T})}|_{\mathbb{W}\setminus\{b_\alpha\}})^{-1}_{xy} \widetilde{G}^{(\mathbb{T})}_{yc_\alpha} - G^{(\mathbb{T}b_\alpha)}_{c_\alpha x} (G^{(\mathbb{T}b_\alpha)}|_{\mathbb{W}\setminus\{b_\alpha\}})^{-1}_{xy} G^{(\mathbb{T}b_\alpha)}_{yc_\alpha} \right).
\end{aligned}$$

The first difference is exactly 0. Therefore, it is sufficient to show the remaining terms are sufficiently small. To do this, we fix x, y and take

$$G^{(\mathbb{T}b_\alpha)}_{c_\alpha x} (G^{(\mathbb{T}b_\alpha)}|_{\mathbb{W}\setminus\{b_\alpha\}})^{-1}_{xy} G^{(\mathbb{T}b_\alpha)}_{yc_\alpha}.$$

We claim that regardless of x, y, the entries of the inverse do not blow up. Therefore, we compare

$$G^{(\mathbb{T}b_\alpha)} - P,$$

where

$$P := G(\operatorname{Ext}(\mathcal{B}_r(\mathbb{W} \setminus \{b_\alpha\}, \mathcal{G}^{(\mathbb{T})}), m_{sc}(z)), z).$$

We can expand $(G^{(\mathbb{T}b_\alpha)} - P)$ according to the Schur complement of each. By the entrywise approximation, we can write

$$(G^{(\mathbb{T}b_\alpha)}|_{\mathbb{W}\setminus\{b_\alpha\}})^{-1} \approx (P^{(\mathbb{T}b_\alpha)}|_{\mathbb{W}\setminus\{b_\alpha\}})^{-1}.$$

By the definition of the switching set, the righthand side is a diagonal matrix, where diagonal entries are order 1. Therefore, $(G^{(\mathbb{T}b_\alpha)}|_{\mathbb{W}\setminus\{b_\alpha\}})^{-1}$ is a matrix with diagonal entries approximately $1/m_d(z)$ and off-diagonal entries of order ε. We can then upper bound this as

$$\sum_{x \in \mathbb{W}\setminus\{b_\alpha\}} |G^{(\mathbb{T}b_\alpha)}_{c_\alpha x}|^2 + \varepsilon \sum_{x \neq y} |G_{xc_\alpha} G_{yc_\alpha}| \leqslant \left((d-1)^\ell + \varepsilon (d-1)^{2\ell} \right) \varphi^2,$$

as the first term dominates. □

In the general version, we make r (the depth of the extension) and ℓ the depth of the switch, different. We then wish to obtain a bound on the entries of the Schur complement of $P^{(\mathbb{T})}$, for which we use the following proposition.

Proposition 3.9. *Assume that $\mathcal{G}$ has excess at most 1, and $\mathcal{H}$ is an induced subgraph such that $B_r(\{i,j\},\mathcal{G}) \subset \mathcal{H}$. Then*

$$|G(\mathrm{Ext}(\mathcal{G},\Delta))_{ij} - G(\mathrm{Ext}(\mathcal{H},\Delta))_{ij}| \lesssim \sqrt{\kappa+\eta}|\Delta - m_{sc}(z)| + |\Delta - m_{sc}(z)|^2 + \left(\frac{|m_{sc}(z)|}{\sqrt{d-1}}\right)^{2r}.$$

Proof of Proposition 3.9. We prove this through the Schur complement formula. The key is to take the Schur complement on vertex set $\mathcal{H}$ for the first graph $\mathcal{G}$. We first realize that the graph of the two are the same on $\mathcal{H}$. The question therefore becomes a question of extension.

After deleting the vertex set $\mathcal{H}$, the graph $\mathcal{G}^{(\mathcal{H})}$ is in many disconnected components. To reason about the boundary we call the boundary edges $\{l_\alpha, a_\alpha\}_{\alpha\in[\![\mu]\!]}$ as before. To consider the Schur complement, we split

$$\mathrm{Ext}(\mathcal{G},\Delta) = \begin{bmatrix} H|_{\mathcal{H}} & B' \\ B & D \end{bmatrix}.$$

Given this, we can take

$$G(\mathrm{Ext}(\mathcal{G},\Delta)) = P, \qquad G(\mathrm{Ext}(\mathcal{H},\Delta)) = P'.$$

We then have the expansion

$$\begin{aligned}
&P_{ij} - P'_{ij} \\
&= \left(-z + H|_{\mathcal{H}} - B'(D-z)^{-1}B\right)^{-1}_{ij} - \left(-z + H|_{\mathcal{H}} - \frac{1}{d-1}B\Delta\mathbb{I}B'\right)^{-1}_{ij} \\
&= \frac{1}{d-1}\sum_{\alpha} P_{il_\alpha}\left((D-z)^{-1}_{a_\alpha a_\alpha} - \Delta\right)P'_{l_\alpha j} \\
&\quad + \frac{1}{d-1}\sum_{\alpha\neq\beta} P_{il_\alpha}(D-z)^{-1}_{a_\alpha a_\beta}P'_{l_\beta j}.
\end{aligned}$$

The first term is small by the fact that diagonal terms are approximated by their self-consistent equations. The second term is small by the fact that we are tangle free: off-diagonal boundary interactions require connections through the exterior, and the excess-at-most-one assumption limits the possible problematic connections. All remaining terms decay like the tree Green's function, giving the error

$$\left(\frac{|m_{sc}(z)|}{\sqrt{d-1}}\right)^{2r}.$$

□

The error term we see here is one of the most striking ways that the random regular graph differs from a matrix model. Namely, when we pass to small neighborhoods, there are low, but not negligible, probability events that create error terms that are hard to deal with. Such error terms are necessary when we consider all graphs. We can see such an error term is generated when there is a cycle in $\mathcal{G}$ that does not exist in $\mathcal{H}$.

4 Iterated Switching

Definition 4.1. *We denote by $\Omega \subset \overline{\Omega}$ the set of d-regular graphs for which the inequalities of Theorem 1.16 hold.*

In order to show universality, we must get a much sharper error term than $N^{-\mathfrak{c}/100}$. To do this, we perform an *iterated switching*. Specifically, we start by wishing to control

$$\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)(Q - Y_\ell(Q))^p]$$

for some $p \geqslant 0$.

By the vertex transitivity of our model, we can write this as

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)(Q - Y_\ell(Q))^p] \\
&= \frac{N}{d}\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)A_{oi}(G^{(i)}_{oo} - Y_\ell(Q))(Q - Y_\ell(Q))^{p-1}] \\
&= \frac{N}{d}\mathbb{E}[\mathbf{1}(\widetilde{\mathcal{G}} \in \Omega)A_{oi}(\widetilde{G}^{(i)}_{oo} - Y_\ell(\widetilde{Q}))(\widetilde{Q} - Y_\ell(\widetilde{Q}))^{p-1}] \\
&= \frac{N}{d}\mathbb{E}[\mathbf{1}(\mathcal{G}, \widetilde{\mathcal{G}} \in \Omega)A_{oi}(\widetilde{G}^{(i)}_{oo} - Y_\ell(Q))(Q - Y_\ell(Q))^{p-1}] + \text{error}
\end{aligned}$$

doing similar calculations as before. Once again as before, we can expand this with the resolvent identity. What we saw is that in expectation, the first term

$$(d-1)^\ell \frac{N}{d}\mathbb{E}[\mathbf{1}(\mathcal{G}, \widetilde{\mathcal{G}} \in \Omega)A_{oi}P^2_{ol_\alpha}(G^{(b_\alpha)}_{c_\alpha c_\alpha} - Q)(Q - Y_\ell(Q))^{p-1}]$$

has expectation 0 when we take the switch. Therefore, when we do the expansion, we have the next term

$$(d-1)^\ell \frac{N}{d}\mathbb{E}[\mathbf{1}(\mathcal{G}, \widetilde{\mathcal{G}} \in \Omega)A_{oi}P^2_{ol_\alpha}(G^{(b_\alpha)}_{c_\alpha c_\alpha} - Q)^2 R].$$

For the previous bound, we could just bound this as ϵ^2, but now we want to do something stronger. Therefore we *iterate* the sampling, by doing a new sampling, this time around c_α. Note that when we did this previous sampling we picked up a new term, meaning the infinity norm bound has decreased. Therefore, our overall process is to resample around a vertex for which there is a diagonal term. If we can resample $\mathfrak{b}$ times for some sufficiently large constant $\mathfrak{b}$, then as ϵ is polynomial, we can make this overall error term N^{-2}, which is always sufficiently small.

The tricky thing is that we need to do expansion in all terms, and cannot just immediately pass to a Ward identity every time. One of the keys to the universal expansion is to do the Woodbury formula.

4.1 Switching using the Woodbury formula

Last time we showed how to use the Schur complement can be used to bound error, but now we want to show how to give a sharper bound. This is crucial to show universality. Instead of the Schur complement, we use a different expansion.

Let $A + UCV^\top$ be a rank r perturbation of A. Namely, $U, V \in \mathbb{R}^{N\times r}$ and $C \in \mathbb{R}^{r\times r}$. Then, the Woodbury formula gives us

$$(A + UCV^\top)^{-1} - A^{-1} = -A^{-1}U(C^{-1} + V^\top A^{-1}U)^{-1}V^\top A^{-1}. \tag{4.1}$$

In this section, we introduce a novel expansion based on the Woodbury formula (4.1). In the rest of this section, we assume that $|\mathsf{W}| = \mu$ Then the switching edges $(b_\alpha, c_\alpha)_{\alpha\in[\![\mu]\!]}$ are far away from each other, and have large tree neighborhood.

We compare the normalized adjacency matrix of the switched graph to that of the original graph, $\widetilde{H} - H$. We recall

$$\widetilde{H} - H = -\sum_{\alpha \in [\![\mu]\!]} \xi_\alpha, \quad \xi_\alpha := \frac{1}{\sqrt{d-1}} \left(\Delta_{l_\alpha a_\alpha} + \Delta_{b_\alpha c_\alpha} - \Delta_{l_\alpha c_\alpha} - \Delta_{a_\alpha b_\alpha} \right).$$

We denote the rank of this difference as $r = \mathrm{O}((d-1)^\ell)$, and rewrite

$$\widetilde{H} - H = UV^\top,$$

where U, V are $N \times r$ matrices, and their nonzero rows correspond to the vertices $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\![\mu]\!]}$. Then, the Woodbury formula (4.1) gives us

$$\widetilde{G} - G = (H - z + UV^\top)^{-1} - (H - z)^{-1} = -GU(\mathbb{I} + V^\top GU)^{-1} V^\top G. \tag{4.2}$$

We now consider the subgraphs that are used in the switch.

$$\mathcal{F} := \mathcal{B}_\ell(o, \mathcal{G}) \cup \bigcup_{\alpha=1}^{\mu} \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}, \quad \widetilde{\mathcal{F}} := \mathcal{B}_\ell(o, \mathcal{G}) \cup \bigcup_{\alpha=1}^{\mu} \{(l_\alpha, c_\alpha), (a_\alpha, b_\alpha)\}.$$

We view $\mathcal{F}, \widetilde{\mathcal{F}}$ as subgraphs of $\mathcal{G}, \widetilde{\mathcal{G}}$ respectively. We will analyze (4.2) using local Green's functions

$$P = G(\mathrm{Ext}(\mathcal{F}, m_{sc}(z)), z), \quad \widetilde{P} = G(\mathrm{Ext}(\widetilde{\mathcal{F}}, m_{sc}(z)), z), \tag{4.3}$$

Both P and $\widetilde{P}$ are simply the Green's function of copies of d-regular trees.

Notice that when restricted to the vertex set of $\mathcal{F}^+$ (which contains the vertices $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\![\mu]\!]}$),

$$\widetilde{P}^{-1} - P^{-1} = \widetilde{H} - H = -\sum_{\alpha \in [\![\mu]\!]} \xi_\alpha = UV^\top. \tag{4.4}$$

We can use the Woodbury formula on $P, \widetilde{P}$ as well, giving

$$\widetilde{P} - P = -PU(\mathbb{I} + V^\top PU)^{-1} V^\top P. \tag{4.5}$$

A crucial observation is that the quantity $-U(\mathbb{I} + V^\top PU)^{-1} V^\top$ in (4.2) and (4.5) take very simple form.

Lemma 4.2. *We introduce the following matrix F, which is nonzero on the vertex set $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\![\mu]\!]}$,*

$$F := \sum_{\alpha \in [\![\mu]\!]} \xi_\alpha + \sum_{\alpha, \beta \in [\![\mu]\!]} \xi_\alpha \widetilde{P} \xi_\beta. \tag{4.6}$$

Then

$$F = -U(\mathbb{I} + V^\top PU)^{-1} V^\top. \tag{4.7}$$

Proof of Theorem 4.2. The nonzero rows of U, V are parametrized by $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\![\mu]\!]}$. By rearranging the above expression (4.5) (we view all the matrices as restricted on the vertex set of $\mathcal{F}^+$), we get

$$P^{-1} \widetilde{P} P^{-1} - P^{-1} = -U(\mathbb{I} + V^\top PU)^{-1} V^\top. \tag{4.8}$$

We can reorganize (4.8) as

$$\begin{aligned} &-U(\mathbb{I} + V^\top PU)^{-1} V^\top = P^{-1} \widetilde{P} P^{-1} - P^{-1} = P^{-1} \widetilde{P} \widetilde{P}^{-1} + P^{-1} \widetilde{P} (P^{-1} - \widetilde{P}^{-1}) - P^{-1} \\ &= P^{-1} \widetilde{P} (P^{-1} - \widetilde{P}^{-1}) = (P^{-1} - \widetilde{P}^{-1}) \widetilde{P} (P^{-1} - \widetilde{P}^{-1}) + \widetilde{P}^{-1} \widetilde{P} (P^{-1} - \widetilde{P}^{-1}) \\ &= (P^{-1} - \widetilde{P}^{-1}) + (P^{-1} - \widetilde{P}^{-1}) \widetilde{P} (P^{-1} - \widetilde{P}^{-1}) = \sum_{\alpha \in [\![\mu]\!]} \xi_\alpha + \sum_{\alpha, \beta \in [\![\mu]\!]} \xi_\alpha \widetilde{P} \xi_\beta = F, \end{aligned} \tag{4.9}$$

where in the last statement we used (4.4). □

Our next lemma attempts to expand $\widetilde{G} - G$ in terms of $\widetilde{P} - P$.

Lemma 4.3. *We assume that $\mathcal{G}, \widetilde{\mathcal{G}} \in \Omega$ and take $G^\circ = (G - P)$. Then we have:*

$$\widetilde{G} - G = \sum_{k \geqslant 0} GF(G^\circ F)^k G, \tag{4.10}$$

Proof. Thanks to (3.4), $\max_{x,y \in \mathbb{S}}\{|G^\circ_{xy}|\} \leqslant \varepsilon$ uniformly for $x, y \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\![\mu]\!]}$. We can then expand (4.2) using the resolvent identity and (4.7) to conclude that

$$\begin{aligned}
\widetilde{G} - G &= -GU(\mathbb{I} + V^\top GU)^{-1} V^\top G = -GU(\mathbb{I} + V^\top PU + V^\top G^\circ U)^{-1} V^\top G \\
&= -GU \left((\mathbb{I} + V^\top PU)^{-1} \sum_{k \geqslant 0} (-1)^k (V^\top G^\circ U (\mathbb{I} + V^\top PU)^{-1})^k \right) V^\top G \\
&= \sum_{k \geqslant 0} (-1)^{k+1} GU(\mathbb{I} + V^\top PU)^{-1} (V^\top G^\circ U (\mathbb{I} + V^\top PU)^{-1})^k V^\top G \\
&= \sum_{k \geqslant 0} GF(G^\circ F)^k G.
\end{aligned}$$

This gives (4.10). □

We now show how to use this calculation on our above term. We write

$$\begin{aligned}
\widetilde{Q} - Q &= \frac{1}{Nd} \sum_{u \sim v} (\widetilde{G}^{(v)}_{uu} - G^{(v)}_{uu}) \\
&= \frac{1}{Nd} \sum_{u \sim v} (\widetilde{G}_{uu} - \widetilde{G}_{uv} (\widetilde{G}_{vv})^{-1} \widetilde{G}_{vu} - G_{uu} + G_{uv} (G_{vv})^{-1} G_{vu}).
\end{aligned}$$

We then want to get rid of the inverse term. Therefore, we do the expansion

$$(G_{vv})^{-1} = \sum_{k \geqslant 0} \frac{(m_d(z) - G_{vv})^k}{m_d(z)^{k+1}}.$$

Therefore, by the given entrywise bounds, we can write the above as

$$= \frac{1}{Nd} \sum_{u \sim v} (\widetilde{G}_{uu} - \widetilde{G}_{uv} \left(\sum_{k=0}^{\mathfrak{b}} \frac{(m_d(z) - \widetilde{G}_{vv})^k}{m_d(z)^{k+1}} \right) \widetilde{G}_{vu} - G_{uu} + G_{uv} \left(\sum_{k=0}^{\mathfrak{b}} \frac{(m_d(z) - G_{vv})^k}{m_d(z)^{k+1}} \right) G_{vu} + O(N^{-2}).$$

for some sufficiently large constant $\mathfrak{b}$, where we have used the entrywise convergence.

We therefore have reduced to terms of the form

$$\widetilde{G}_{uu} - G_{uu} = \sum_{k=0}^{\mathfrak{b}} (GF((G - P)F)^k G)_{uu} + O(N^{-2}).$$

We take the first term, future terms can be dealt with similarly.

$$(GFG)_{uu} = \sum_{x,y \in \mathbb{S}} G_{ux} F_{xy} G_{yu}.$$

If (x, y) are in different connected components of $\mathcal{F}$, then there is some "free" index, giving an overall expectation of 0. Therefore, we obtain another term around which it is possible to do another resampling argument.

Future terms are smaller than the previous bound by a factor of at least $N^{-\mathfrak{c}/100}$.

References

[1] J. Huang, T. Mckenzie, and H.-T. Yau. Ramanujan property and edge universality of random regular graphs. *arXiv preprint arXiv:2412.20263*, 2024.

[2] J. Huang and H.-T. Yau. Spectrum of random d-regular graphs up to the edge. *Communications on Pure and Applied Mathematics*, 77(3):1635–1723, 2024.

[3] J. Huang and H.-T. Yau. Lecture notes on edge universality for random regular graphs. *arXiv preprint arXiv:2602.00975*, 2026.

[4] A. W. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families I: Bipartite Ramanujan graphs of all degrees. *Annals of Mathematics. Second Series*, 182(1):307–325, 2015.

[5] A. W. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families IV: Bipartite Ramanujan graphs of all sizes. *SIAM Journal on Computing*, 47(6):2488–2509, 2018.

[6] S. J. Miller, T. Novikoff, and A. Sabelli. The distribution of the largest nontrivial eigenvalues in families of random regular graphs. *Experimental Mathematics*, 17(2):231–244, 2008.

[7] P. C. Sarnak. What is... an expander? *Notices of the American Mathematical Society*, 51(7):762–763, 2004.