Typical Support of Closed Walks and Eigenvalue Multiplicity

Theo McKenzie

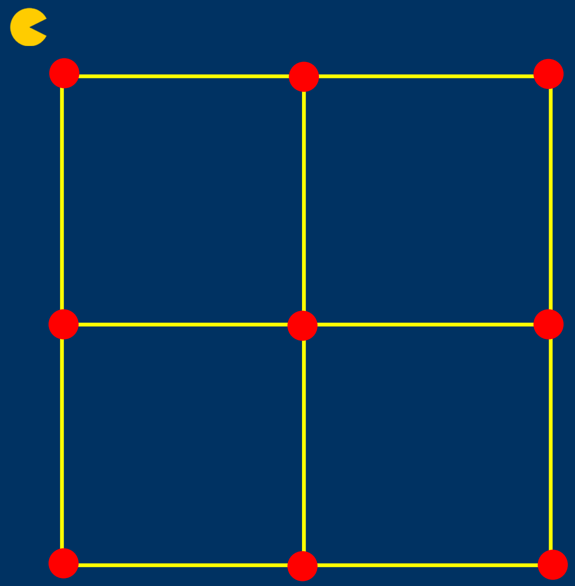
with Peter Rasmussen (University of Copenhagen) and Nikhil Srivastava (University of California, Berkeley)

Delaware Discrete Mathematics/Algebra Seminar
2/24/2021

Setup

- Our goal is to understand the behavior of walks in large graphs.

- A walk is performed by choosing an adjacent node in the graph.



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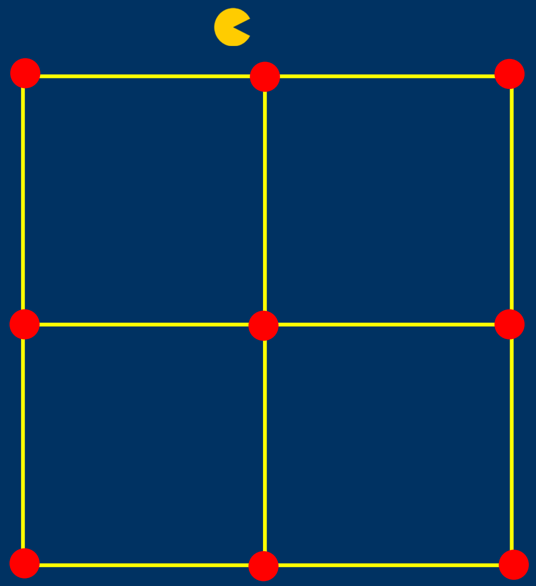

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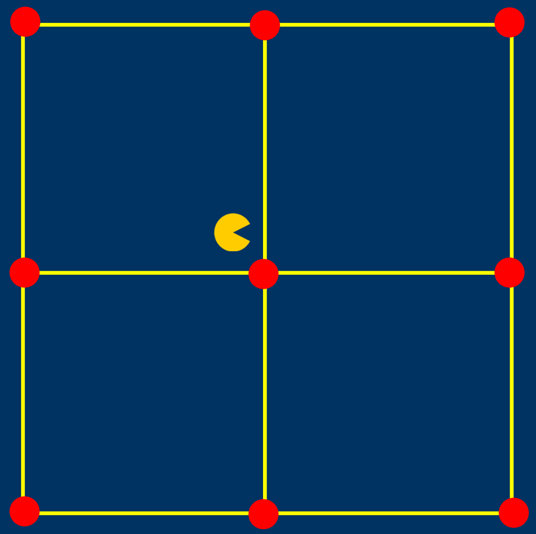

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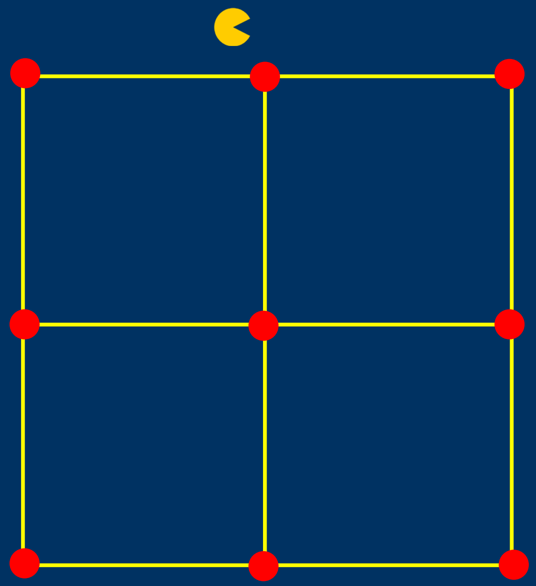

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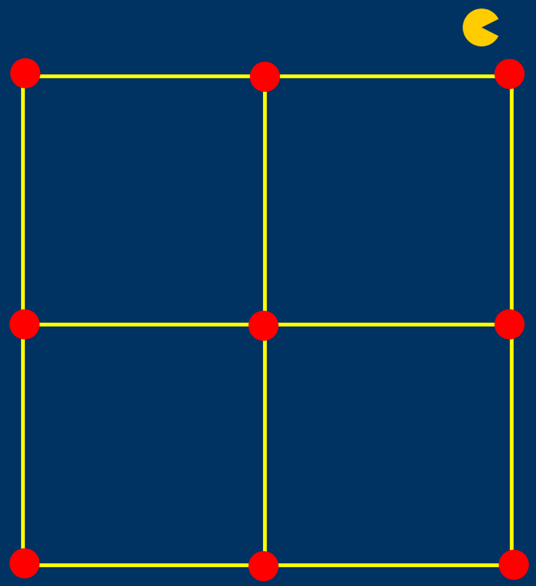
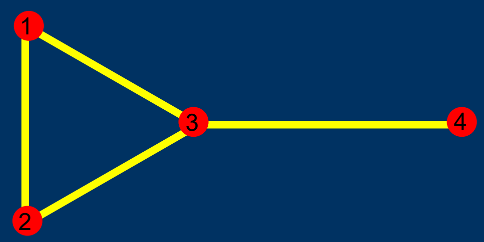


Adjacency Matrix

- Encode the walk through an "adjacency matrix" A, with rows/columns corresponding to the vertices, and putting a 1 between connected vertices.



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Note that as the matrix is symmetric, the eigenvalues are real and can be ordered $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, where n is the number of vertices.
- Multiplying by the matrix can be thought of as a step in the walk.
- The entry $\left(A^k\right)_{uv}$ corresponds to walks of length k between u and v.

Equiangular Lines

Question (van Lint and Seidel 1966):

What is the maximum number of lines in $\mathbb{R}^d$ that all share the same angle θ, for $0 < \theta < \pi/2$?

Theorem (Jiang, Tidor, Yao, Zhang and Zhao '19):
If there exists a minimal k such that there exists a graph on k vertices with spectral radius exactly $(1-\alpha)/(2\alpha)$, then the maximum is $\lfloor k(d-1)/(k-1) \rfloor$ for $\alpha = \arccos(\theta)$ and large enough d.
Otherwise, the maximum is $d + o(d)$.

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Equiangular Lines

The key ingredient was a property of graphs. Namely, showing that for every bounded degree matrix, the multiplicity of the second eigenvalue is $o(n)$.

Theorem (Jiang et al. '19):

For every bounded degree connected graph with n vertices, the second eigenvalue of the adjacency matrix has multiplicity $O(\frac{n}{\operatorname{loglog} n})$.

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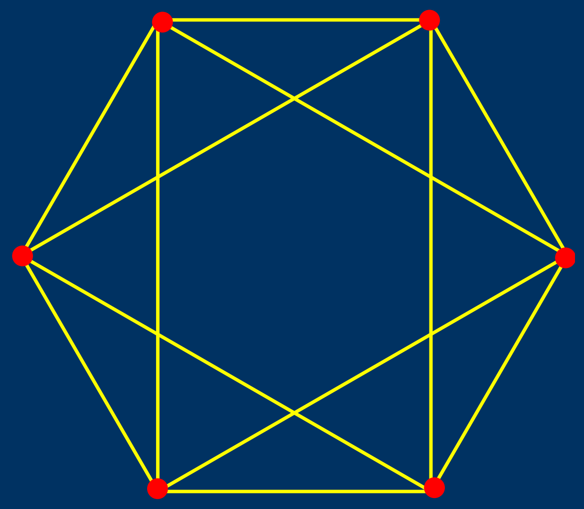
Compare this to the best lower bound, which says that the second eigenvalue of Cayley graphs of $\mathrm{PSL}(2, \mathbb{Z}_p)$ has multiplicity $\Omega(n^{1/3})$.

Theorem

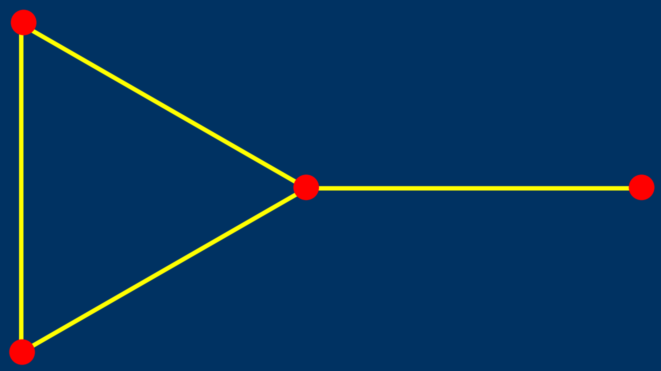
- **Question**: Can we close this gap at all?

- **Theorem A** (M.-Rasmussen-Srivastava): For any bounded degree **regular** connected graph with n vertices, the multiplicity of the second eigenvalue of A is at most $O(\frac{n}{\log^{1/5-o_n(1)} n})$.

Regularity



regular



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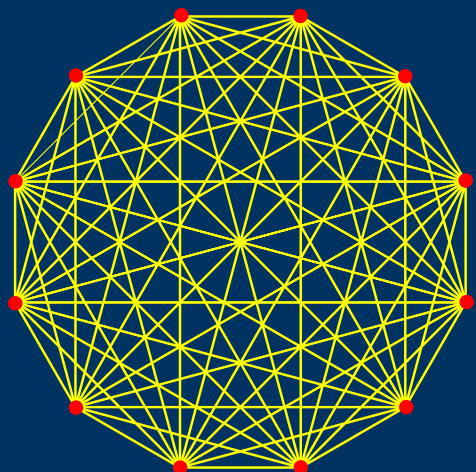
Bounded Degree

There are examples of graphs with non-bounded degree that have high eigenvalue multiplicity.

- For example, the complete graph where every vertex is connected to every other vertex.



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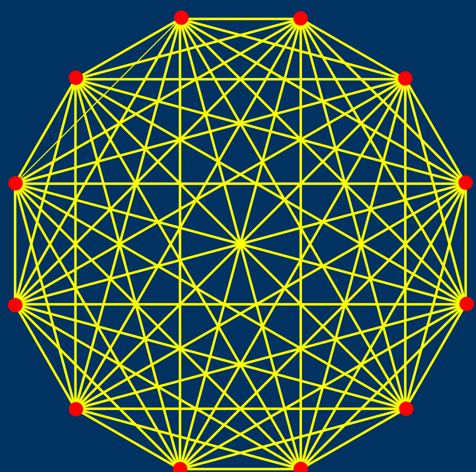

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If J is the all-ones matrix and I the identity, then $A = J - I$. The second eigenvalue has multiplicity $n - 1$.

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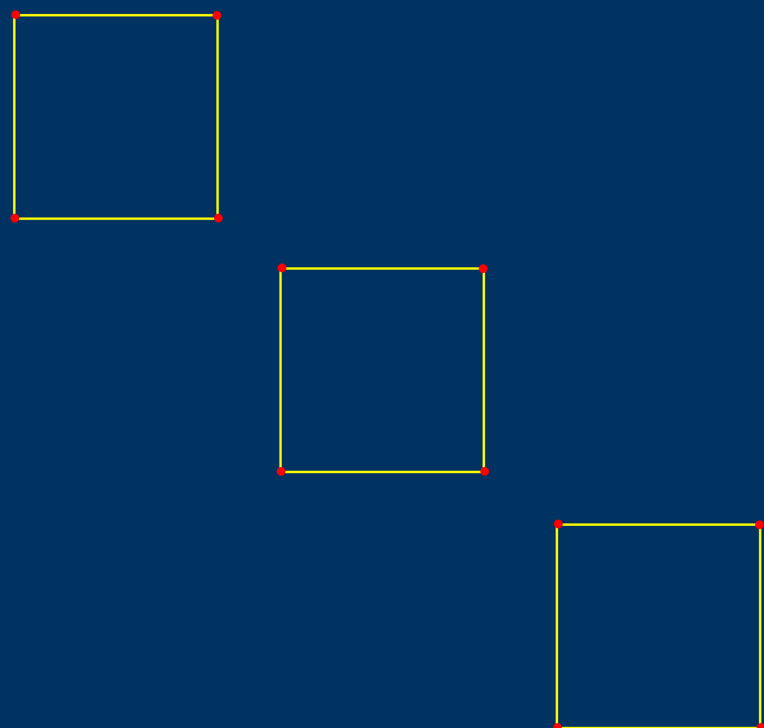
Connected



$$\begin{bmatrix} U & 0 & 0 & \cdots \\ 0 & U & 0 & \cdots \\ 0 & 0 & U & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

If there are k copies of an adjacency matrix U, then the multiplicity of the second eigenvalue is at least k.

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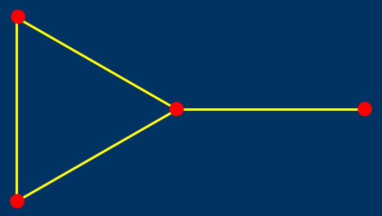
Second eigenvalue

Arbitrary eigenvalues can have linear multiplicity.



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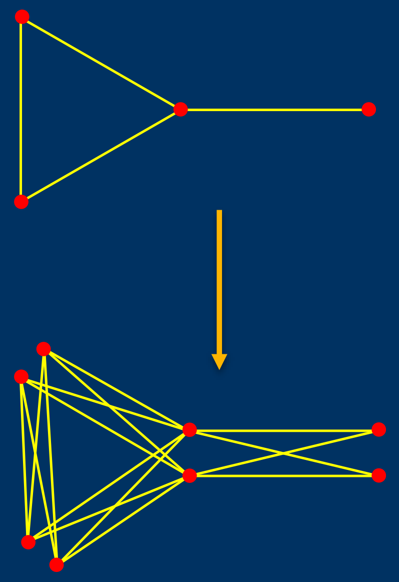

By creating two copies of each vertex, at least half of the eigenvalues are 0.

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Method

- $\text{trace}\left(A^{2k}\right) = \sum_u \left(A^{2k}\right)_{uu} = \sum_{i=1}^{n} \lambda_i^{2k}$

- We use the trace as a proxy for the multiplicity of the eigenvalue.

- We therefore need to bound the trace of A^{2k}.

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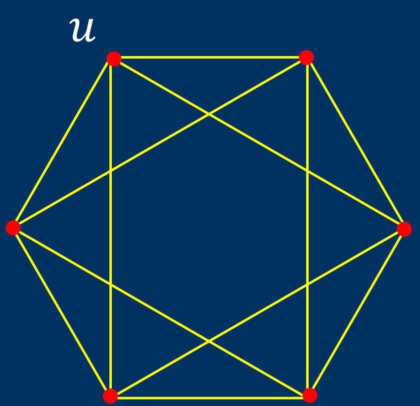
  - The walk starts at u.
  - We walk on our graph for $2k$ steps.
  - The walk ends at u.
- **Definition**: A walk is closed if it ends where it starts.
- Our goal is to count the number of closed walks.

Cauchy's Interlacing Theorem

- **Cauchy's Interlacing Theorem**: for an $n-1 \times n-1$ principal submatrix with eigenvalues μ_i of an $n \times n$ matrix A with eigenvalues λ_i, the eigenvalues interlace: $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$.
- Cauchy's Interlacing Theorem tells us that if the multiplicity of λ_2 in any submatrix of size $n-s$ is m, then in the original graph it is at most $m+s$.
- **Idea:** Delete vertices such that most closed walks are deleted. As there are now few closed walks, then the trace and the multiplicity of λ_2 is low *on this subgraph*. Then the multiplicity of λ_2 in the full graph is that of the subgraph plus the number of vertices deleted.

Vertex Deletion

- The diagonal entries of A^{2k} correspond to the walks that are closed. We want to show most walks are deleted.



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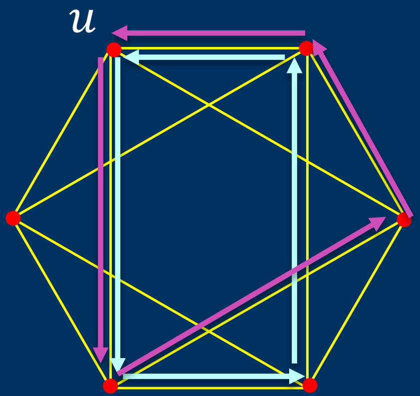

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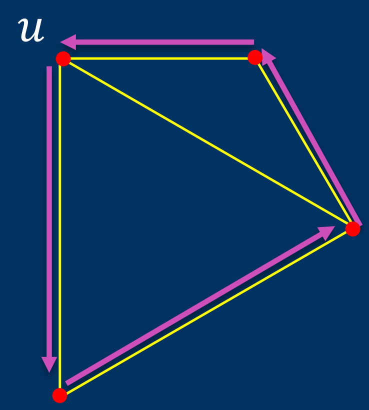

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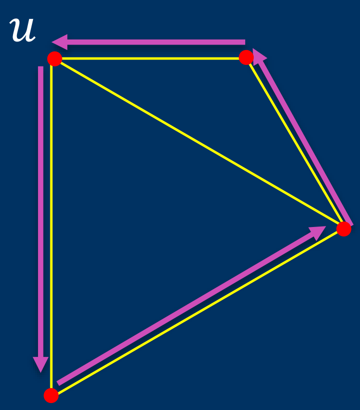

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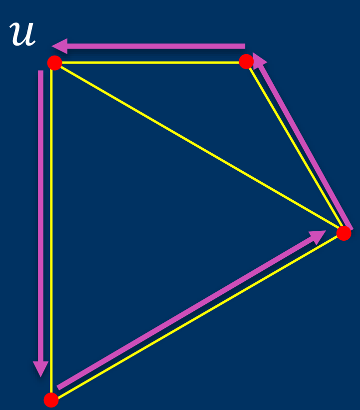
- Most walks will be deleted if most walks hit many different vertices.
- If we delete most walks, then the trace is low, meaning the multiplicity in the subgraph is low, meaning the original multiplicity is low.



Support Question

New Goal: Show that most closed walks of length k visit many vertices.

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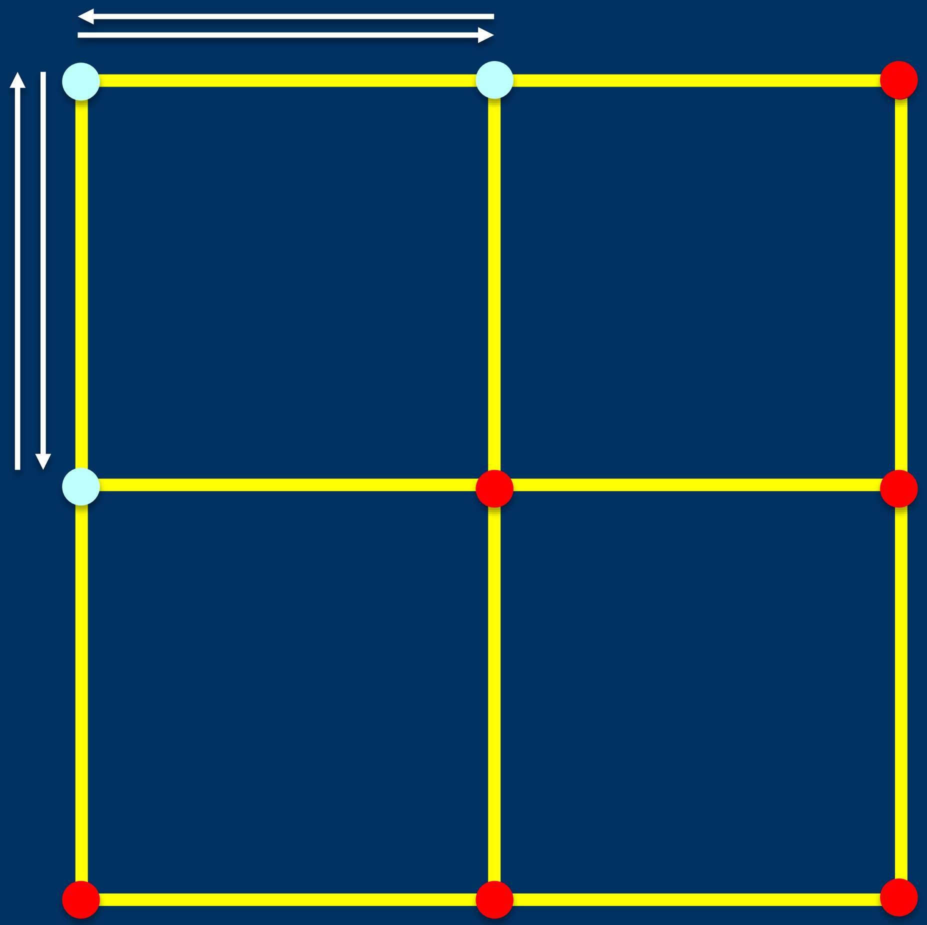
- If I do not care whether the walk is closed, the problem is much easier. The hard part is the closed condition.

Example



High Support

Theorem B: With high probability, the support of a closed walk of length $2k$ on a regular graph has support $\Omega\left(k^{1/5}\right)$.

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Theorem B (detailed):

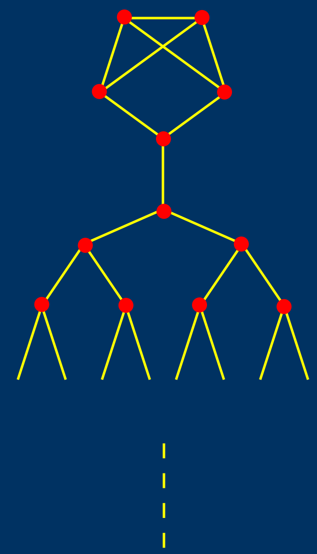
The probability a closed walk of length $2k$ on a regular graph has support at most s is $o_k(1)$ of the probability of having support at most $2s$, for $s = O(k^{1/5})$.

The path (warmup)

- For the path, on average we travel $\Theta(\sqrt{k})$ from the starting vertex on a closed walk of length $2k$.

The bulb tree (warmup)

- For the tree with a bulb at the root, on average we do not travel more than $\Theta(\log k)$ from the bulb.



Walks of Small Support

- How many walks are closed, starting at u and stay within the set S?
- If a walk stays within the set S, then it is counted in the quadratic form of the submatrix $e_u^T A_S^{2k} e_u$.

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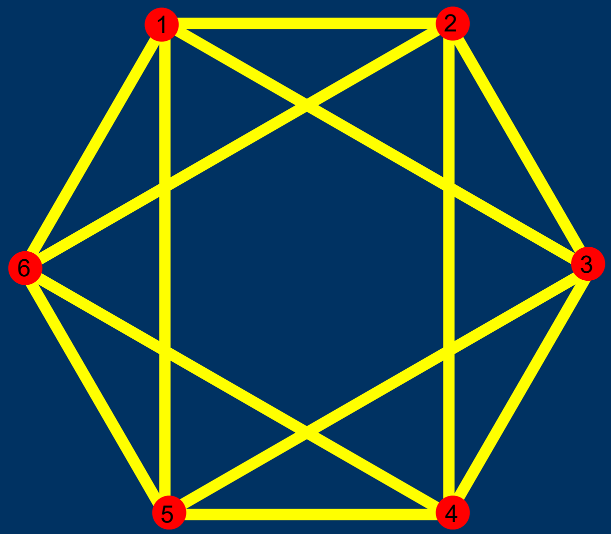
- This quadratic form is upper bounded by the top eigenvalue of A_S, λ_S.



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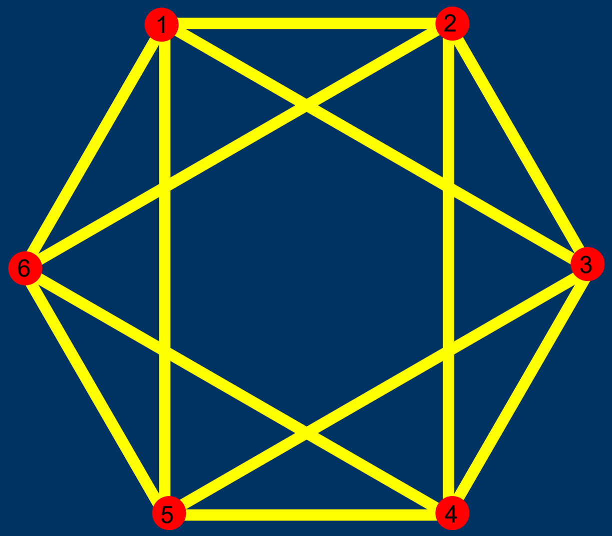

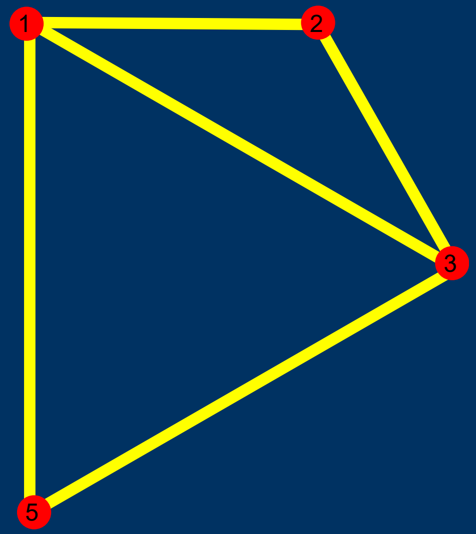

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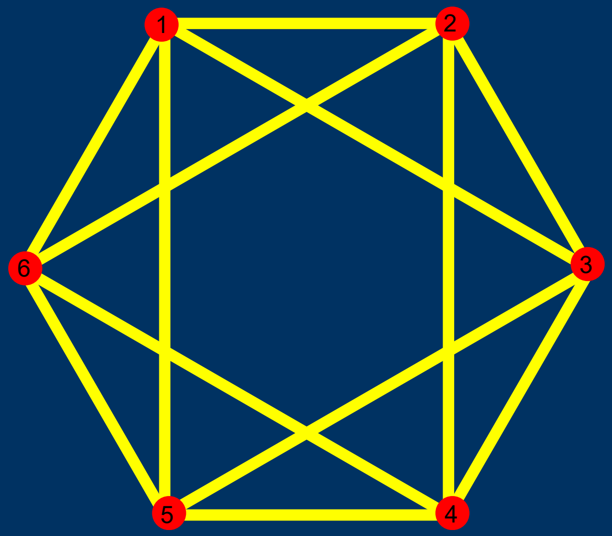

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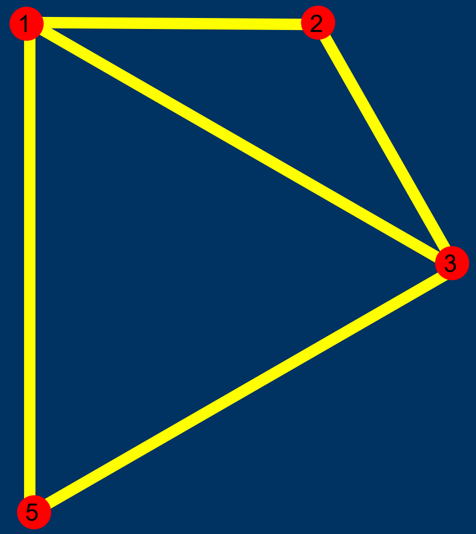

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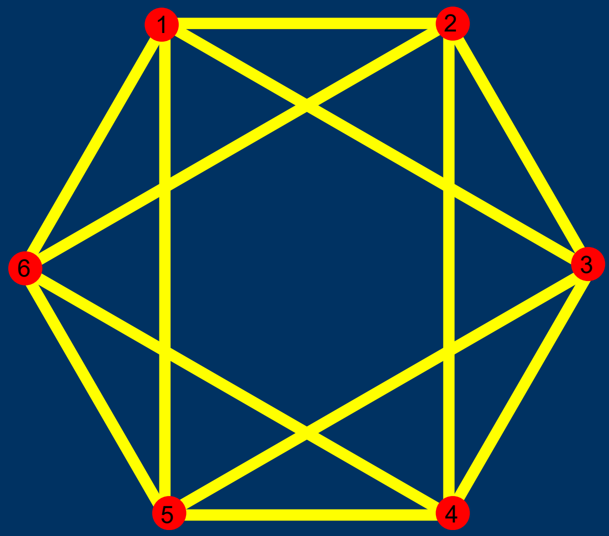

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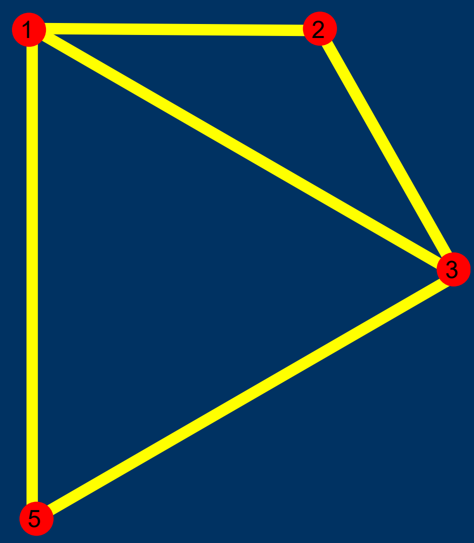

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We can count walks on this support using the submatrix A_S.

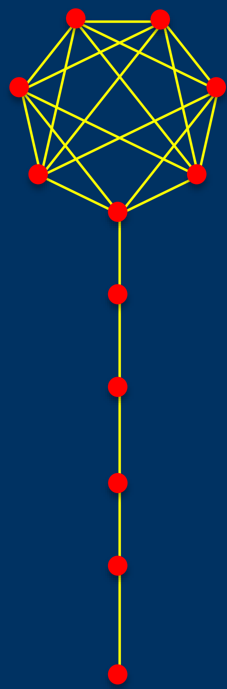
Resolution

- The number of closed walks remaining on a set S after $2k$ steps is at most λ_S^{2k}.
- Moreover, we know that for large k, $e_v^T A_S^{2k} e_v \approx \langle \psi_S, e_v \rangle^2 \lambda_S^{2k}$.
- **Claim**: To show that there are many more walks on T than S, it is sufficient to show that $\lambda_T^{2k} \gg \lambda_S^{2k}$.
- If I find a set T such that $\lambda_T \geq (1+\epsilon)\lambda$, then even if ϵ is small, for large enough k, $\lambda_T^{2k} \gg \lambda_S^{2k}$.
- **Lemma A**: There exists a set $T = S \cup \{v\}$ such that $\lambda_T > \lambda_S(1 + \frac{c}{|S|^5})$. Therefore, for $k \gg |S|^5$, $\lambda_T^{2k} \gg \lambda_S^{2k}$.

How to increase eigenvalue?

- **Lemma B:** For any vertex $v \notin S$ that neighbors a vertex $u \in S$, the top eigenvalue of $T = S \cup \{v\}$ satisfies $\lambda_T \geq \lambda_S + \Omega(\psi_S(u)^2)$.
- **Remark:** As we are looking at a subset of a regular graph, we can extend the graph at any vertex which is not of maximal degree in S.
- **Theorem C**: For any connected graph of bounded degree, there is a vertex u of non-maximal degree such that $\psi(u) = \Omega(n^{-\frac{5}{2}})$.
- Note that we only need one vertex with non-maximal degree to have large value in ψ. Cioabă and Gregory give a lower bound on the minimum value in the principal eigenvector, but that bound is exponentially small.

Example of the Lollipop



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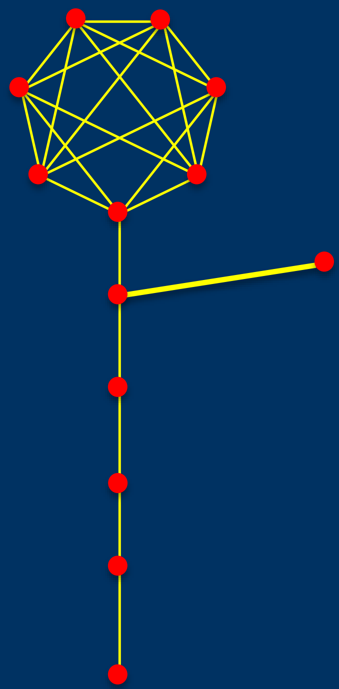

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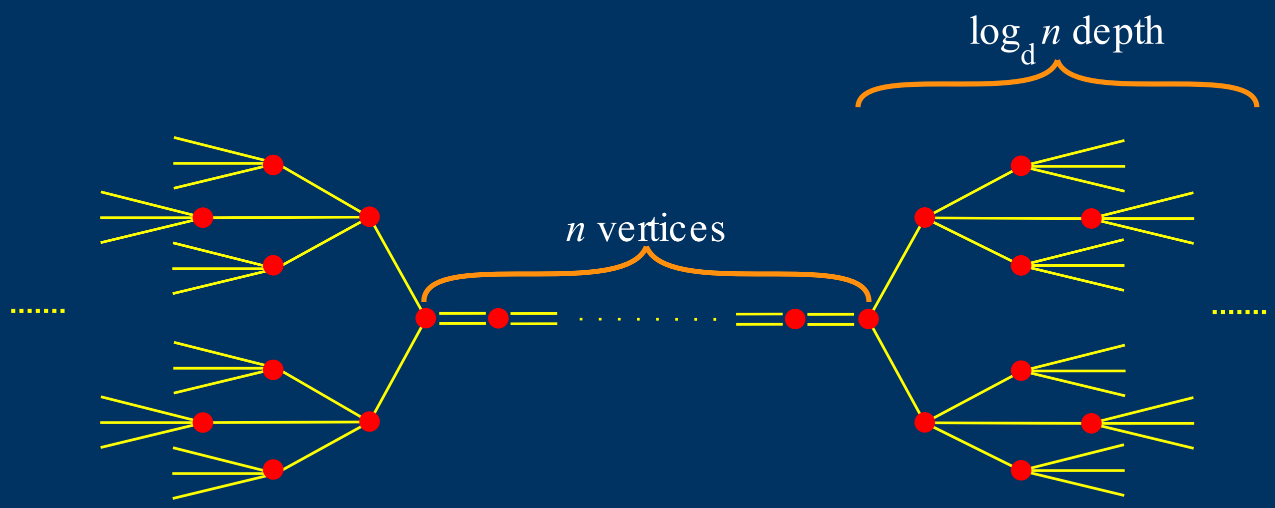
Result

- We know that for some vertex t, we have
$$\psi_S(t) \geq 1/\sqrt{|S|}$$
- If we can bound the ratio
$$\psi_S(u)/\psi_S(t)$$
for this t, that is sufficient.
- **Lemma C**: There exists a u of non-maximal degree for which
$$\frac{\psi_S(u)}{\psi_S(t)} = \Omega(\frac{1}{D|\partial S|})$$
where D is the diameter of the graph, and $|\partial S|$ is the number of vertices of non-maximal degree.
- Both these quantities are at most $|S|$, which translates into a bound of
$$\psi_S(u) \geq 1/|S|^{\frac{5}{2}}$$

Mangrove

- The worst case is when both the boundary and diameter are the order of $|S|$. In the below example, for all u on the boundary $\psi(u) = \Theta(1/|S|^{\frac{5}{2}})$, which is the bound given by our work.



Full Results

- Our full result bounds the number of eigenvalues in an interval.

- **Theorem A** (full): The number of eigenvalues of A in the interval $\left[\left(1-\frac{\log\log n}{\log n}\right)\lambda_2, \lambda_2\right]$ is $O\left(\frac{n}{\log^{1/5-o_n(1)} n}\right)$.

- For bipartite Ramanujan graphs, the number of eigenvalues in this interval is $\Omega\left(\frac{n}{\log^{3/2} n}\right)$, meaning our result is tight except for potentially the exponent.

Full Increase

Lemma A (full): For every connected subset of vertices S of a regular graph there is a subset of vertices $T \supset S$, $|T| = 2|S|$, such that

$$\lambda_1(A_T) \geq \lambda_1(A_S) + \frac{c}{|S|^4}$$

for some constant c.

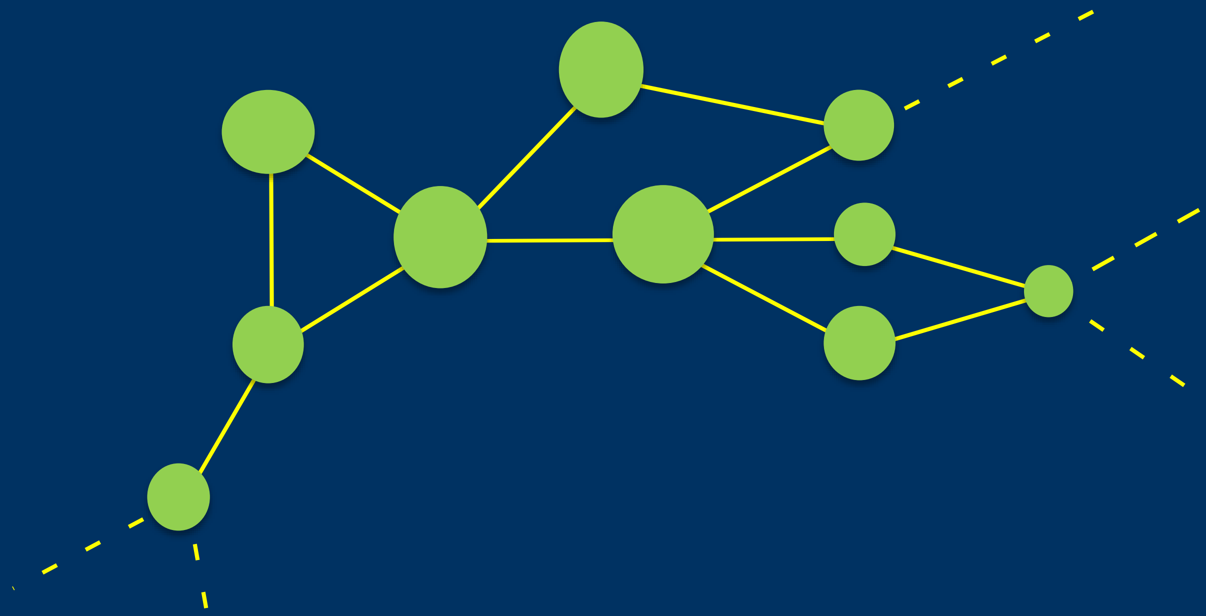
- Method

1) Show $\exists u \in \partial S$ such that $\psi_S(u) \geq \frac{1}{|S|^{5/2}}$
2) Show that for a vertex $v \notin S$ adjacent to u,

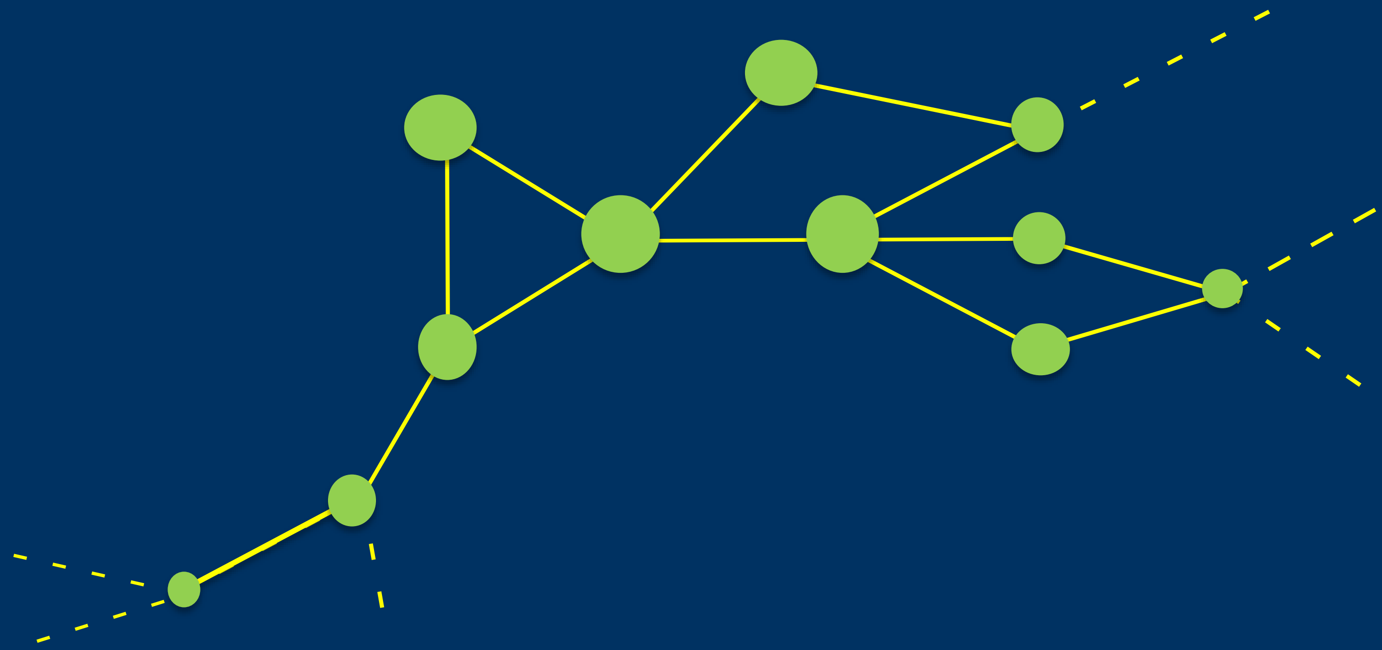
$$\lambda_1(A_{S \cup v}) = \lambda_1(A_S) + \Omega(\psi_S(u)^2).$$

Repeat this process $|S|$ times to achieve a set of size $2|S|$.

Visualization of 2



Visualization of 2

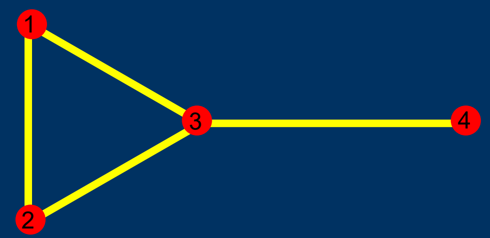


Visualization of 2

Random Walk Matrix

- We can encode the distribution of the random walk through its random walk matrix W, which is such that W_{vu} is the probability of transitioning from node u to node v.

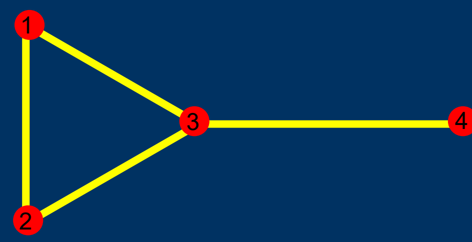
- For a simple random walk, the probability of transitioning between two connected vertices u and v is the reciprocal of the degree of u.



$$\begin{bmatrix} 0 & 1/2 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 1 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}$$

Spectral Theory

The key thing about regular graphs is that their adjacency matrices are the same up to rescaling to the random walk matrix.



Adjacency matrix A

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Walk matrix W

$$\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 1 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}$$

Theorem

We now consider any random walk matrix W of a graph with maximum degree Δ, and the boundary of S ∂S.

Lemma C (full): For any connected subgraph S, there is a vertex $u \in \partial S$ such that $\psi_S(u) \geq \frac{1}{\Delta |S|^2} \psi_S(t)$ for any vertex $t \in S$.

All of our results generalize to the random walk matrix.

Observations

1) Lower bound $\psi_S(u)$ for some $u \in \partial S$

- By the power method, $\lim_{r\to\infty} \frac{W_S^r \mathbf{1}_S}{\|W_S^r \mathbf{1}_S\|}$ approaches ψ. Therefore, $\lim_{r\to\infty} \frac{\mathbf{1}_S^T W_S^r e_u}{\mathbf{1}_S^T W_S^r e_t} = \frac{\psi(u)}{\psi(t)}$.

- $W_S^r e_u$ represents the random walk distribution for the walk remaining on S.

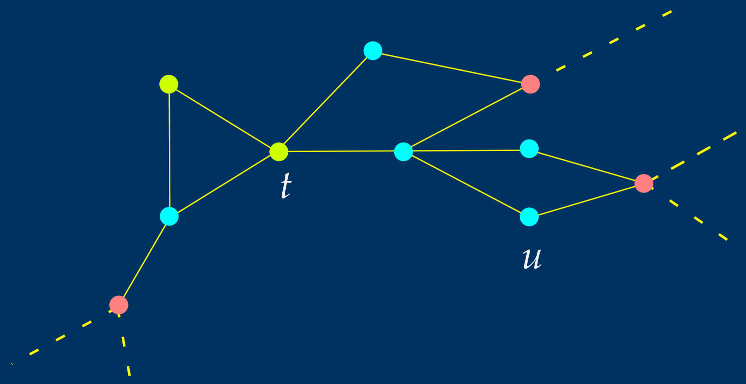
- $\frac{\mathbf{1}_S^T W_S^r e_u}{\mathbf{1}_S^T W_S^r e_t}$ is the ratio of the probabilities of walks starting at u and t remaining in our set for r steps.

- The probability that we remain in the set does not change if we contract all the points directly outside the boundary to a single point s.

Visualization

1) Lower bound $\psi_S(u)$ for some $u \in \partial S$

There's no difference in the probability we remain in the set if we contract all the points immediately outside the boundary to one point.



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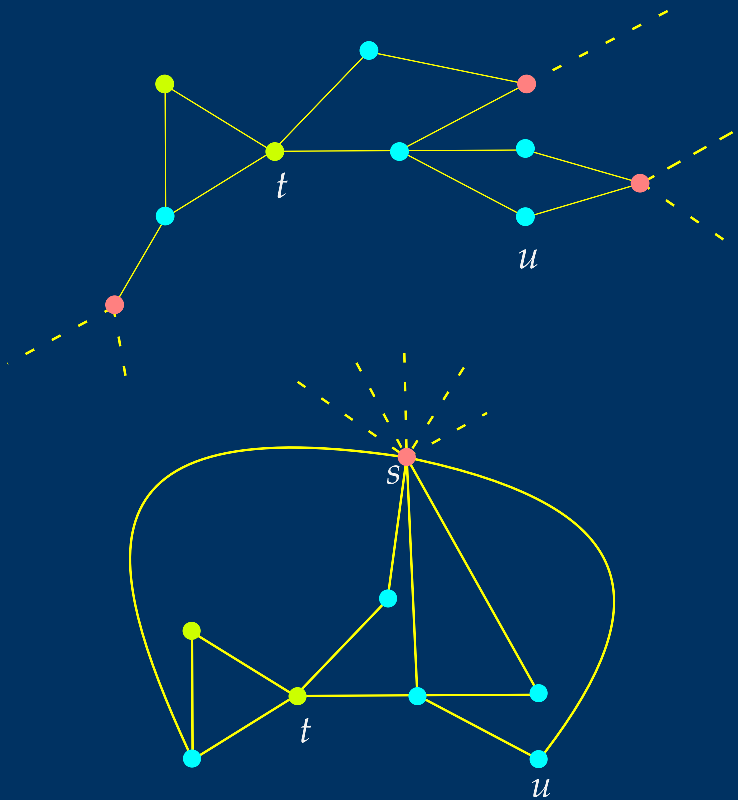

Observations

1) Lower bound $\psi_S(u)$ for some $u \in \partial S$

- If we know the walk reaches t before it reaches s, and it takes j steps to reach t, then the probability that it stays within S is the probability a walk of length $r - j$ starting at t stays within S.

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- Specifically, if Y_j is the event that the walk hits t before s AND hits t for the first time at step j, then

$$\mathbf{1}_S^T W_S^r e_u \geq \sum_{j=0}^{r} \Pr\left(Y_j\right) \mathbf{1}_S^T W_S^{r-j} e_t \geq \mathbf{1}_S^T W_S^r e_t \sum_{j=0}^{r} \Pr(Y_j)$$

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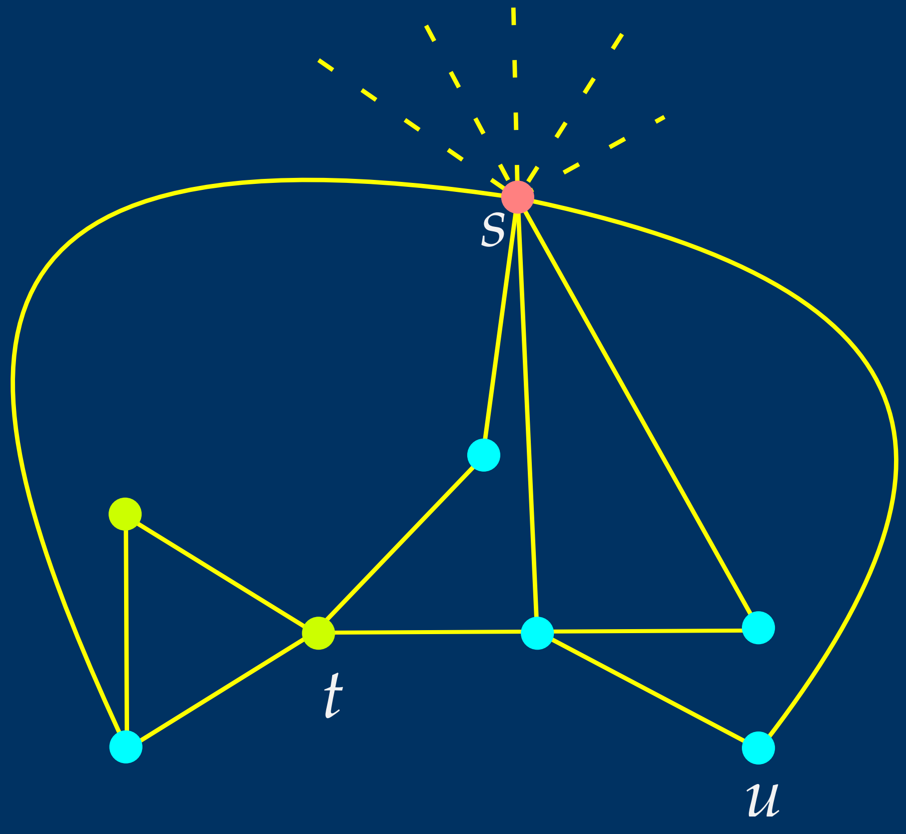
- As the expected hitting time of t is finite, $\lim_{r\to\infty} \left(\sum_{j=0}^{r} \Pr(Y_j)\right)$ is the probability that t is hit before s.

Visualization

1) Lower bound $\psi_S(u)$ for some $u \in \partial S$

A random walk of length r conditioned on reaching t before s that reaches t for the first time at step j has the same probability of staying within the set as a random walk starting at t of length $r - j$.



Electric Flow

1) Lower bound $\psi_S(u)$ for some $u \in \partial S$

- This probability is equivalent to the voltage of u, denoted $V(u)$, in a flow from s to t where the voltages $V(t) = 1, V(s) = 0$ [e.g. Bollobás].

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- By Ohm's law, the total current from s to t is at least $1/dist(s,t)$. Because s has at most $|S|\Delta$ neighbors, there is some neighbor of s such that current through this vertex is at least $1/\Delta|S|^2$. Therefore, the voltage of this vertex is at least $1/\Delta|S|^2$, as is the probability of reaching t before s.

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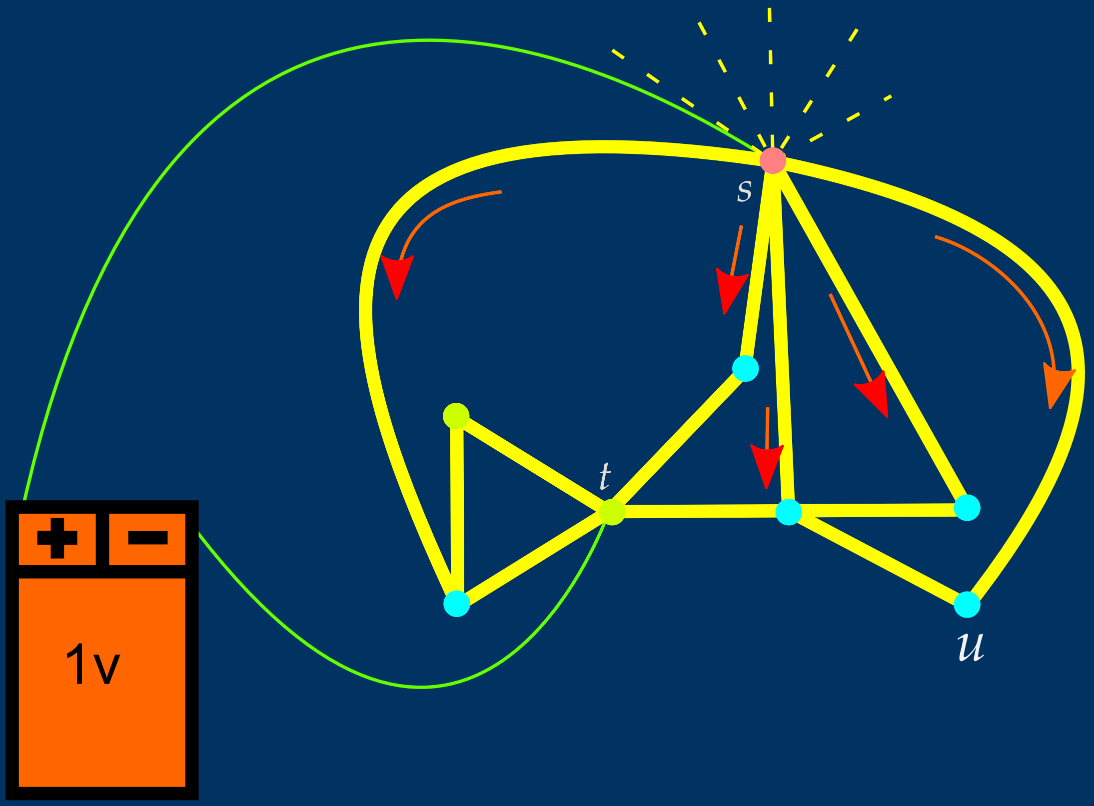
- For some vertex $u \in \partial S$, $\frac{\psi_S(u)}{\psi_S(t)} \geq \frac{1}{\Delta|S|^2}$. As we can assume $\psi_S(t) \geq 1/\sqrt{S}$,

$$\psi_S(u) \geq 1/(\Delta|S|^{5/2}).$$

Electric Flow

1) Lower bound $\psi_S(u)$ for some $u \in \partial S$

We interpret this probability as an electrical current between s and t. There must be one vertex adjacent to s that receives a large current flow and therefore has high eigenvector value. We make this our u.



Questions

- Is $\Omega(k^{1/5})$ tight for the typical support of a walk on a bounded degree graph.

- Is there an $\epsilon > 0$ such that the random walk matrix of every graph has second eigenvalue multiplicity $O(n^{1-\epsilon})$?
  - The most we know is that there are graphs with second eigenvalue multiplicity $\Omega\left(n^{1/3}\right)$.

Thank you!