Many Nodal Domains in Random Regular Graphs

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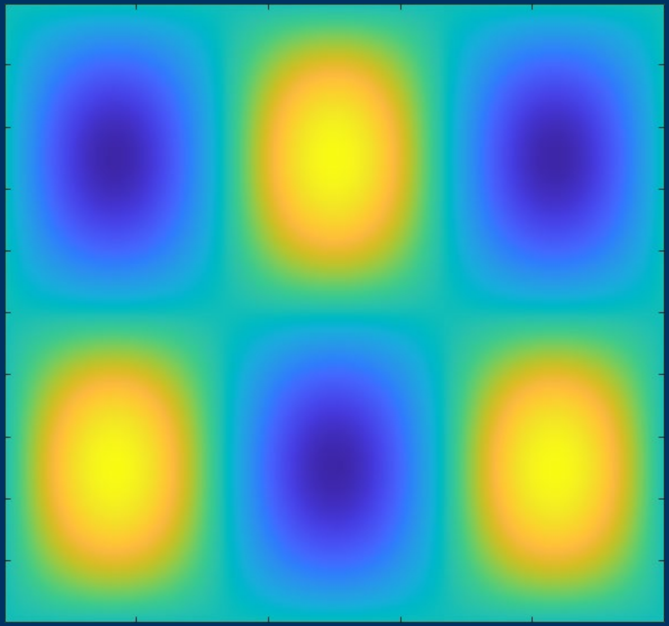
Courant's Nodal Domain Theorem

- [Courant] The zero set of the kth smallest Dirichlet eigenfunction of the Laplacian on a smooth bounded domain in $\mathbb{R}^d$ partitions it into at most k components.
- These components, known as nodal domains, have garnered significant attention in spectral geometry and mathematical physics.



A heat map of the 6th Dirichlet eigenfunction of the square.

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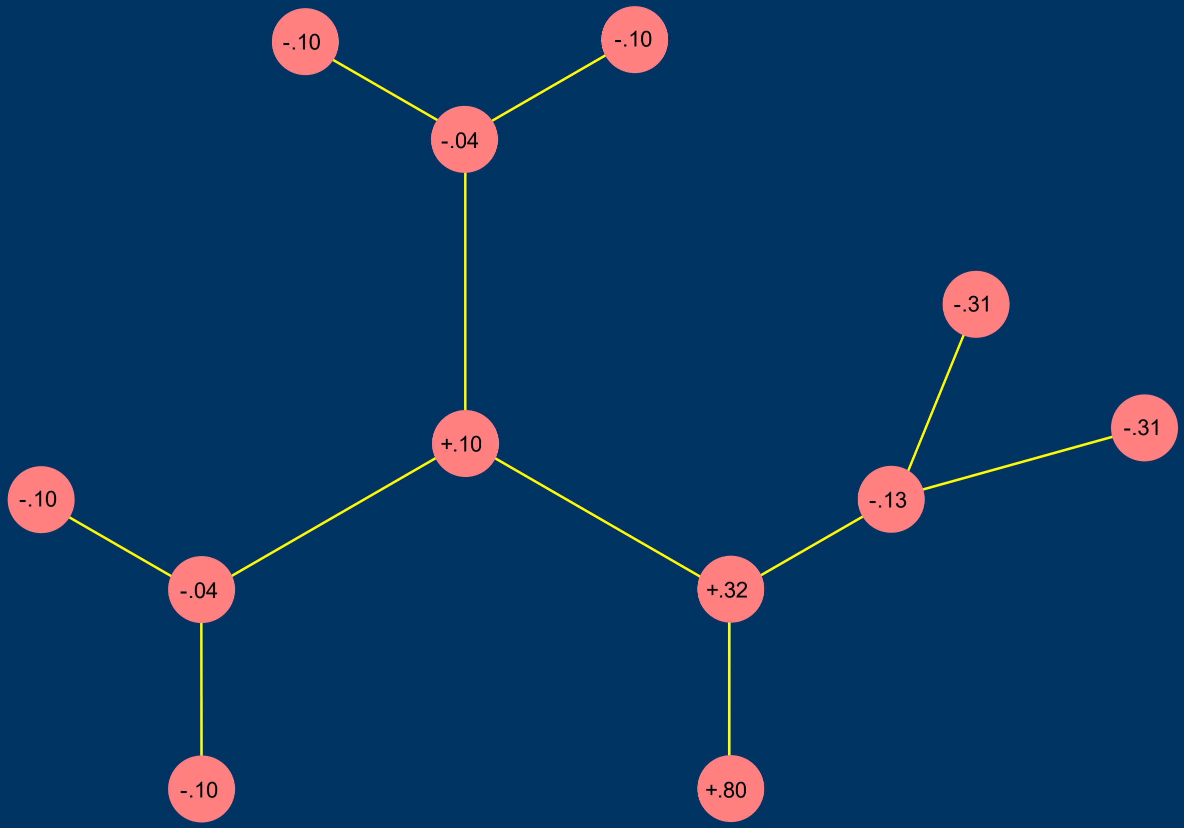

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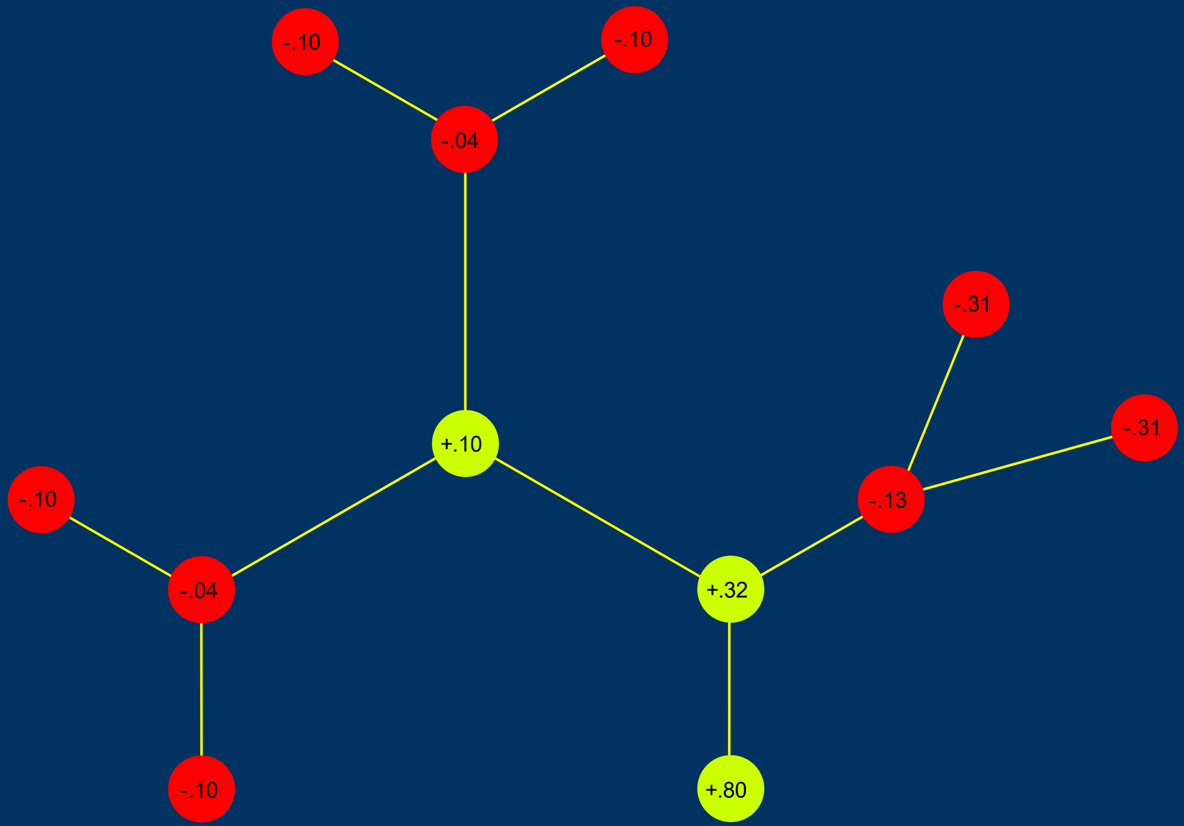

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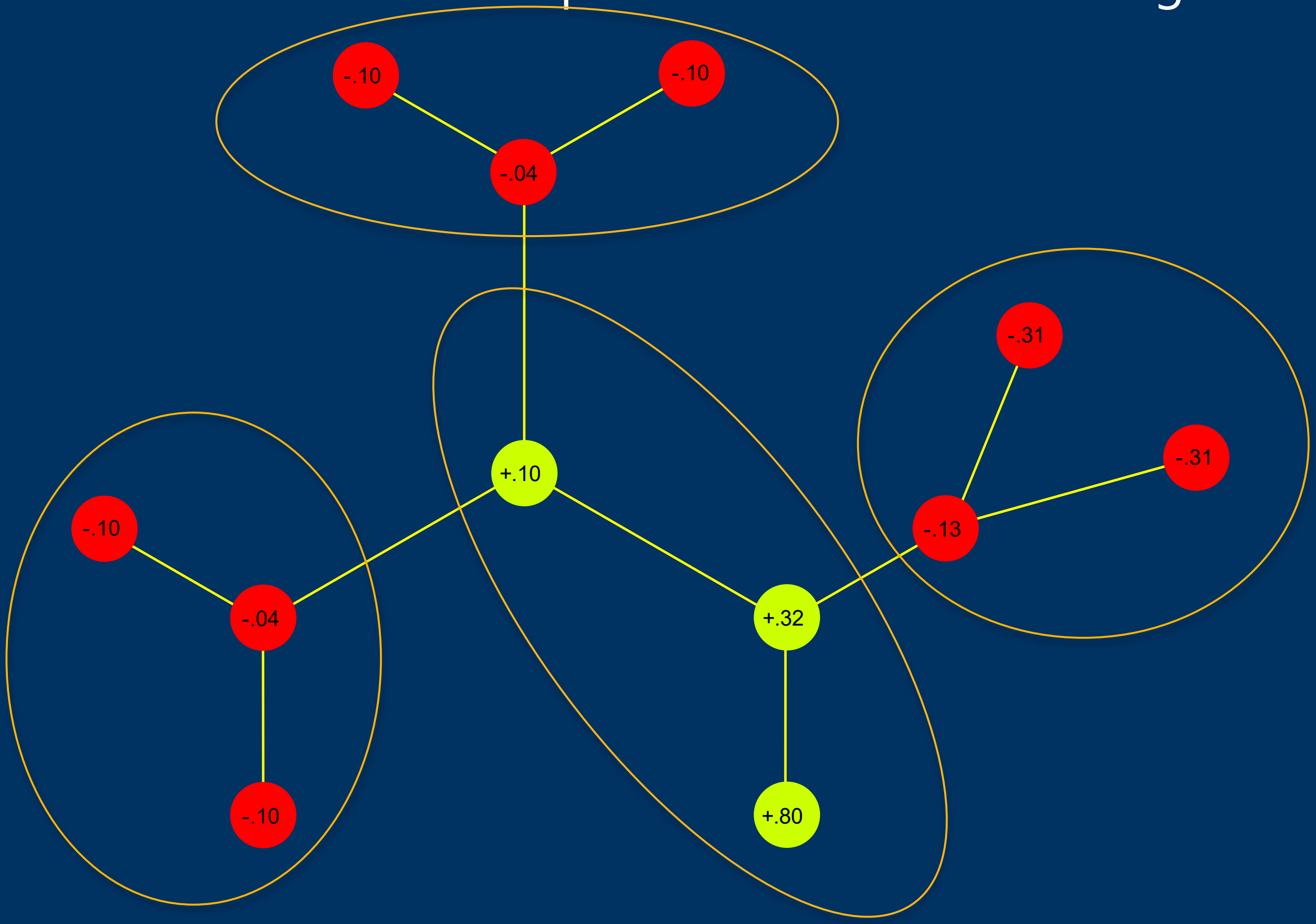

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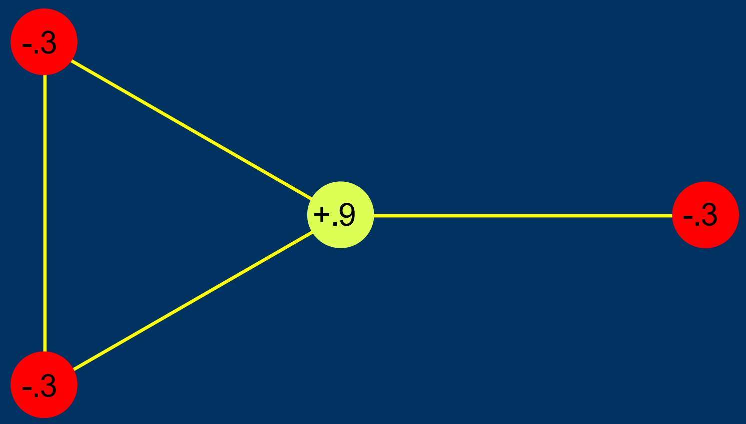

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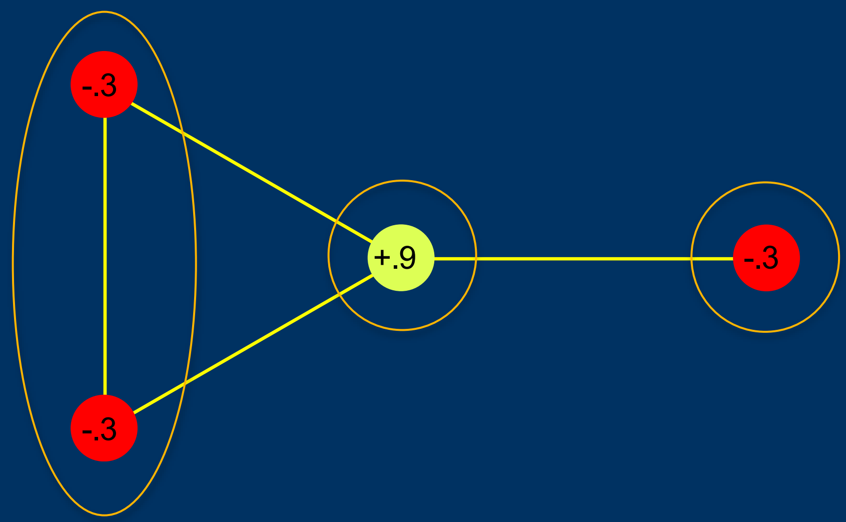
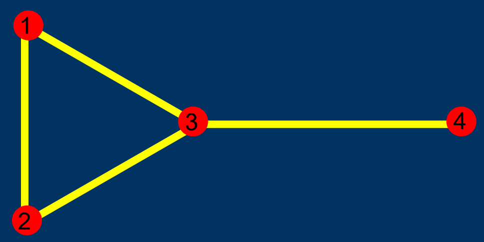


Adjacency Matrix

- Encode graphs through an adjacency matrix A, with rows/columns corresponding to the vertices, and placing a 1 where there is an edge.



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Note that as the matrix is symmetric, the eigenvalues are real.

- We can also consider the combinatorial Laplacian $D - A$, where D is the diagonal matrix of degrees.

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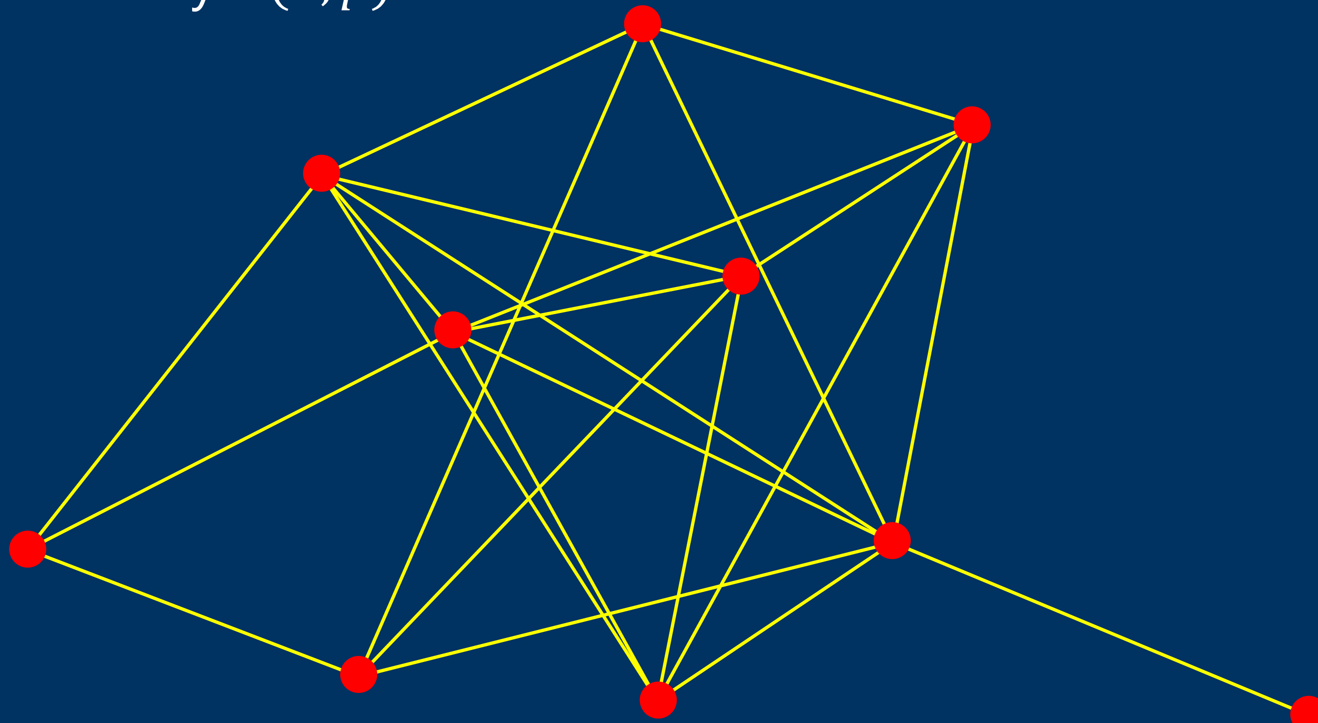
- Question: Can we lower bound the number of nodal domains for graphs that have many edges?
- What about random graphs?

Erdős-Rényi Graphs

- For a graph on n vertices, include each of the $\binom{n}{2}$ potential edges independently with probability p. This distribution is denoted by $G(n, p)$.



Sampled from $G(10{,}0.5)$

Results for Erdős-Rényi Graphs

- [Dekel-Lee-Linial 2011] For a $G(n,p)$ graph for fixed $p \in (0,1)$, with high probability every eigenvector of the **adjacency matrix** besides the first has all but $O(1)$ vertices in two nodal domains.

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- [H. Huang-Rudelson 2020] For bulk eigenvectors with $p \in (n^{-c}, 1)$ and edge eigenvectors for fixed $p \in (0,1)$ the nodal domains are approximately the same size.

Methods for Erdős-Rényi Graphs

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Theorem: There is an $\alpha = \alpha(p, \delta), \beta = \beta(p)$ such that for any fixed vector $\boldsymbol{w} \in \mathbb{R}^m$, and Q any $(1 + \delta)k \times k$ matrix with i.i.d. Bernoulli(p) entries.

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- H. Huang and Rudelson use a result of Bourgade, J. Huang and H.T. Yau:

Theorem: Eigenvectors of Erdős-Rényi graphs exhibit Gaussian behavior.

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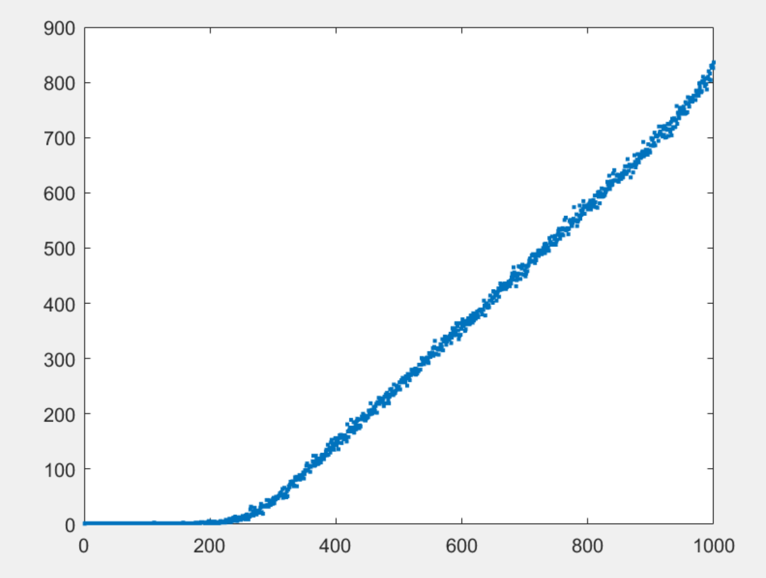

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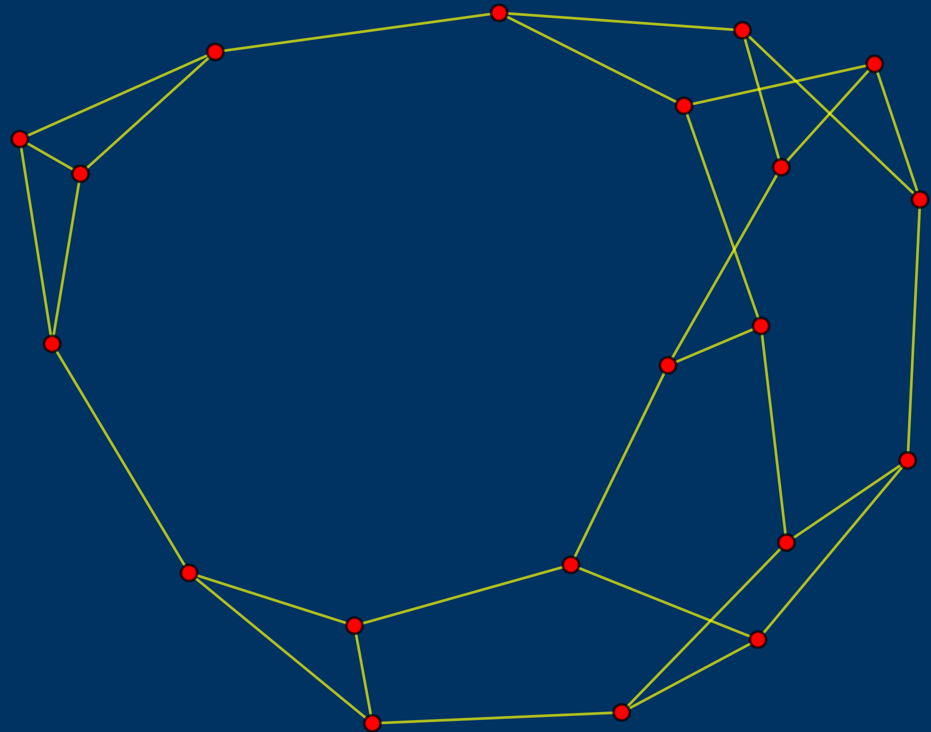
- Dekel, Lee, and Linial also observed that according to simulations, the kth eigenvector of a randomly selected d-regular graph on n vertices has a number of nodal domains that increases with k.
- Our question: can we prove nontrivial structure of nodal domains for a randomly selected d-regular graph, for fixed d.

Random Regular Graphs

- Choose a graph from the set of graphs on n vertices that are d-regular. We denote this distribution by $G(n, d)$.



- Sampled from $G(20,3)$

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- If eigenfunctions have many nodal domains, this would give evidence for Berry's conjecture.

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- [Anantharaman-Le Masson 2015] Prove a statement analogous to Shnirelman's Theorem for large regular graphs.
- Our goal: support the discrete version of Berry's conjecture by showing there are many nodal domains.

Result

[Ganguly-M-Mohanty-Srivastava] Fix $d \geq 3$ and $\alpha > 0$. Then with probability $1 - o_n(1)$, every eigenvector of the adjacency matrix of a $G(n, d)$ sampled graph with eigenvalue $\lambda \leq -2\sqrt{d-2} - \alpha$ has $\Omega(n/\text{polylog}(n))$ nodal domains.

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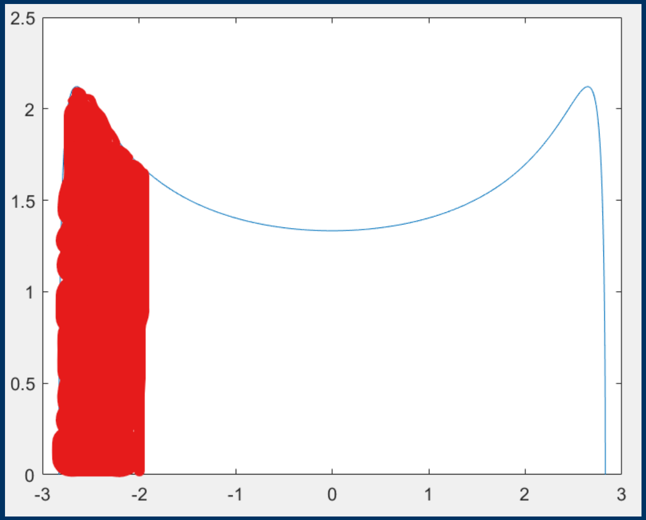

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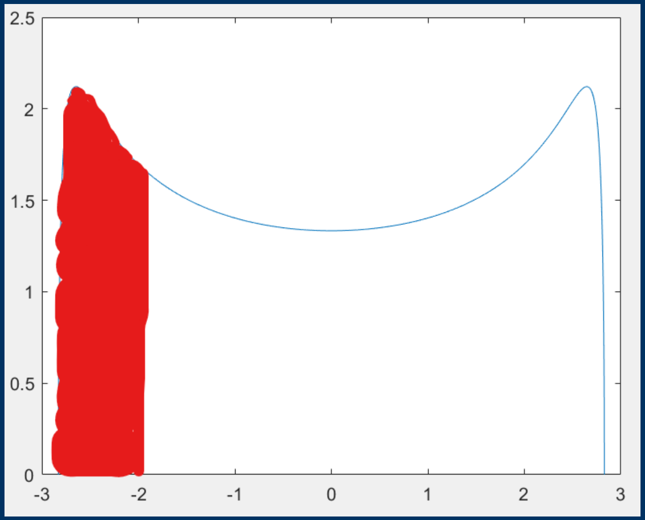

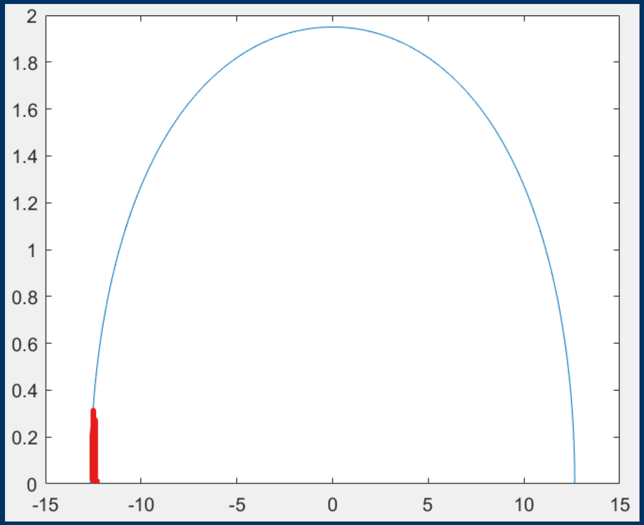

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- For regular graphs, A and the combinatorial Laplacian $L = D - A$ have the same eigenvectors, so we can rephrase the result as concerning the high energy eigenvectors of the Laplacian.

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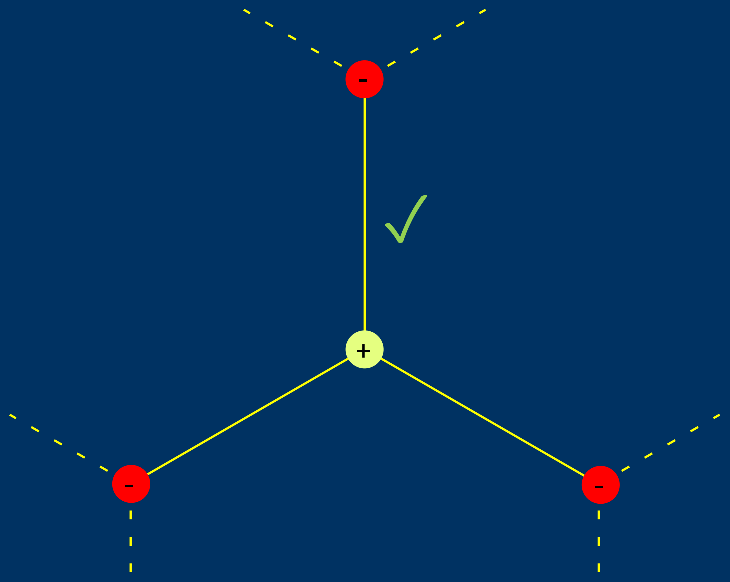

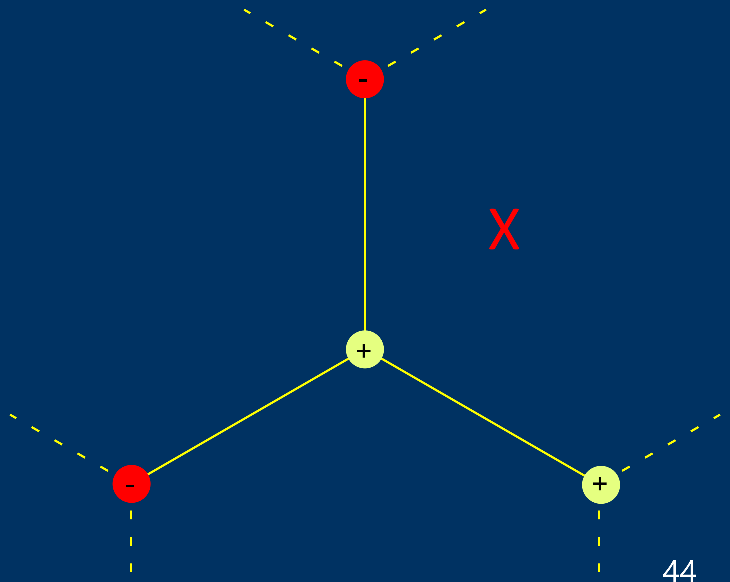

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- We split into cases based on whether the eigenvector is **localized** or **delocalized** (whether the mass of the eigenvector is well spread or not).

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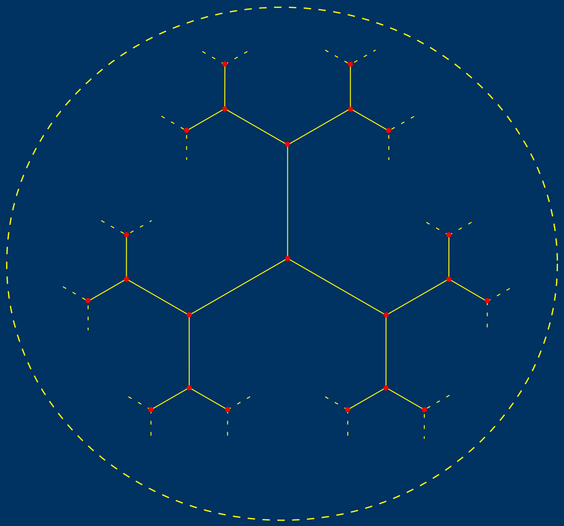

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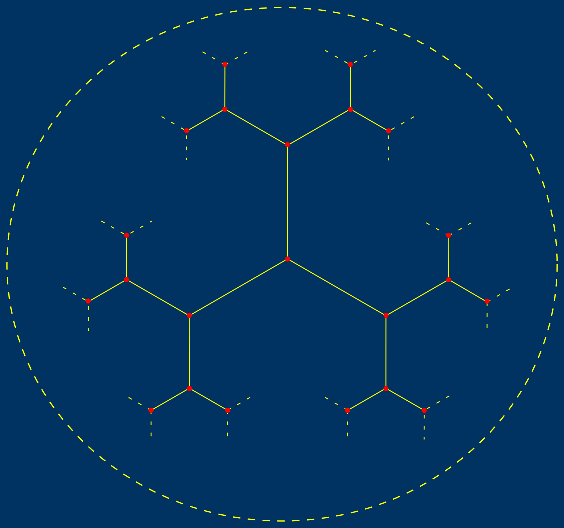

- Therefore, first we can try to understand covariance in the infinite d-regular tree.

The Gaussian Wave

- [Elon 2009] There is a unique joint Gaussian distribution on the vertices of T_d that
  1. Has unit variance in each entry
  2. Is automorphism invariant
  3. Satisfies the eigenvector equation at each vertex.

- This joint Gaussian is called the **Gaussian wave**.

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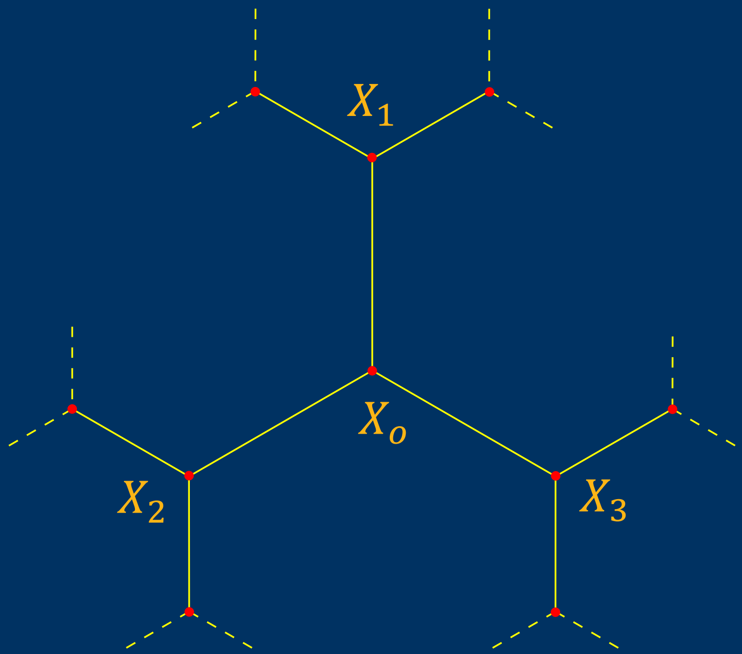

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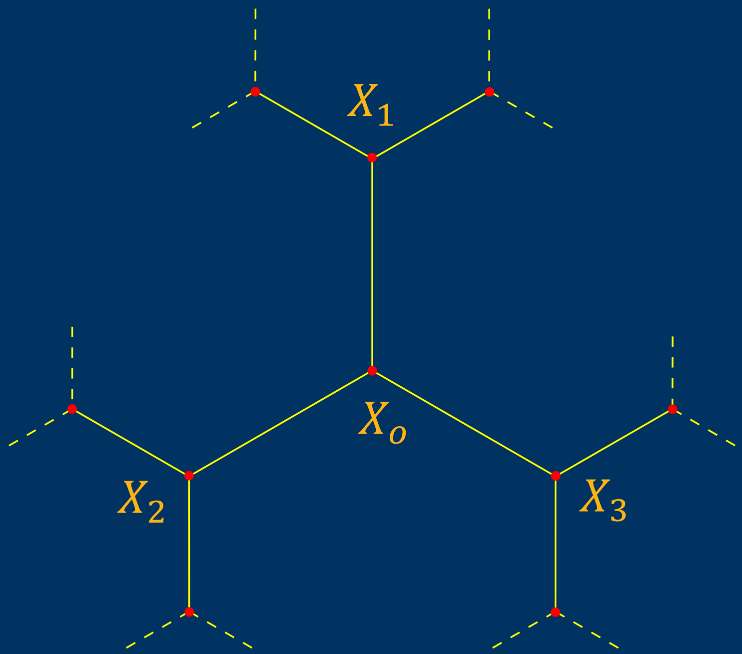

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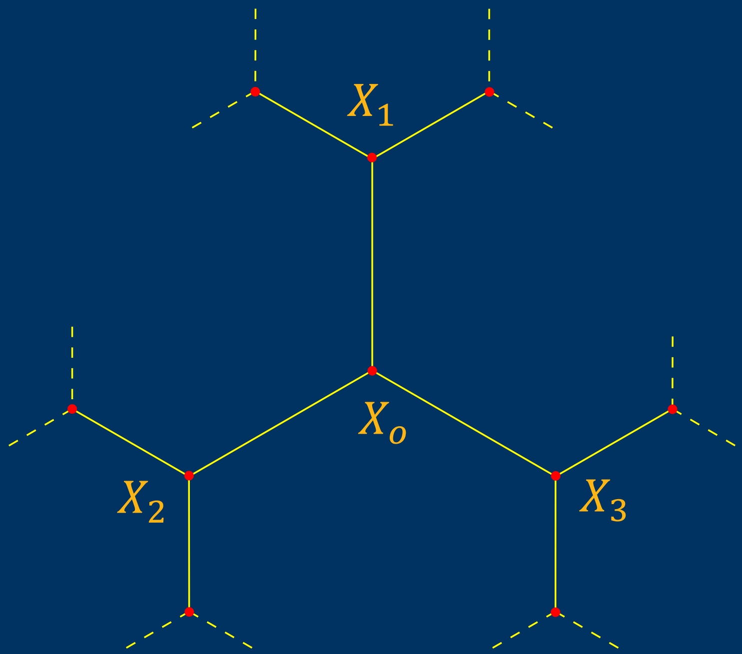

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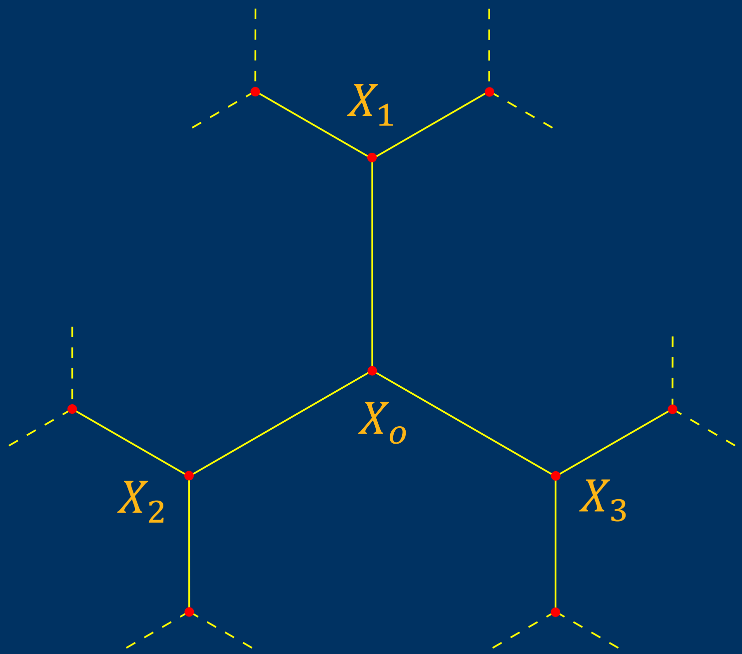

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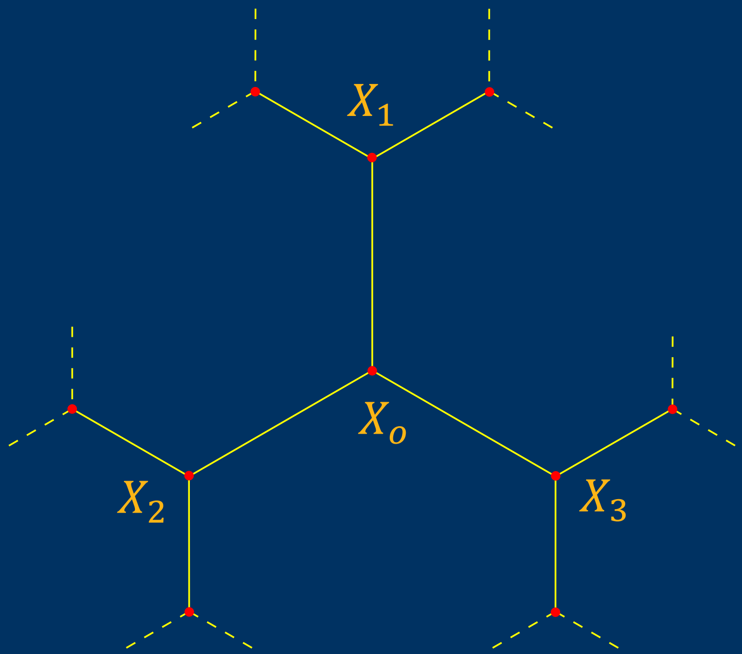

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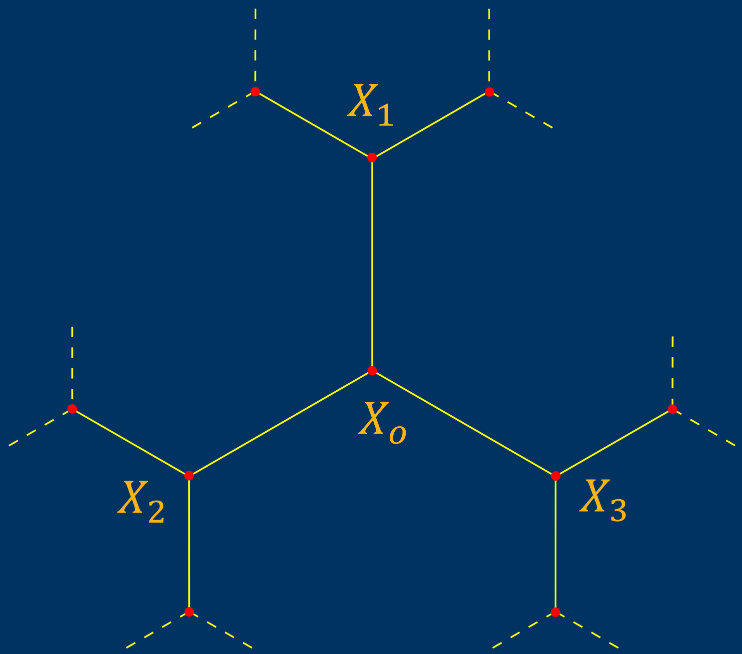
$$\lambda E(X_o^2) = \sum_{v \sim o} E(X_v X_o)$$

$$\lambda = dE(X_1 X_o)$$

Gaussian Wave

1. Has unit variance in each entry
2. Is automorphism invariant
3. Satisfies the eigenvector equation at each vertex



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Convergence to the Gaussian wave

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- [Backhausz-Szegedy 2019] Prove a form of Berry’s conjecture.

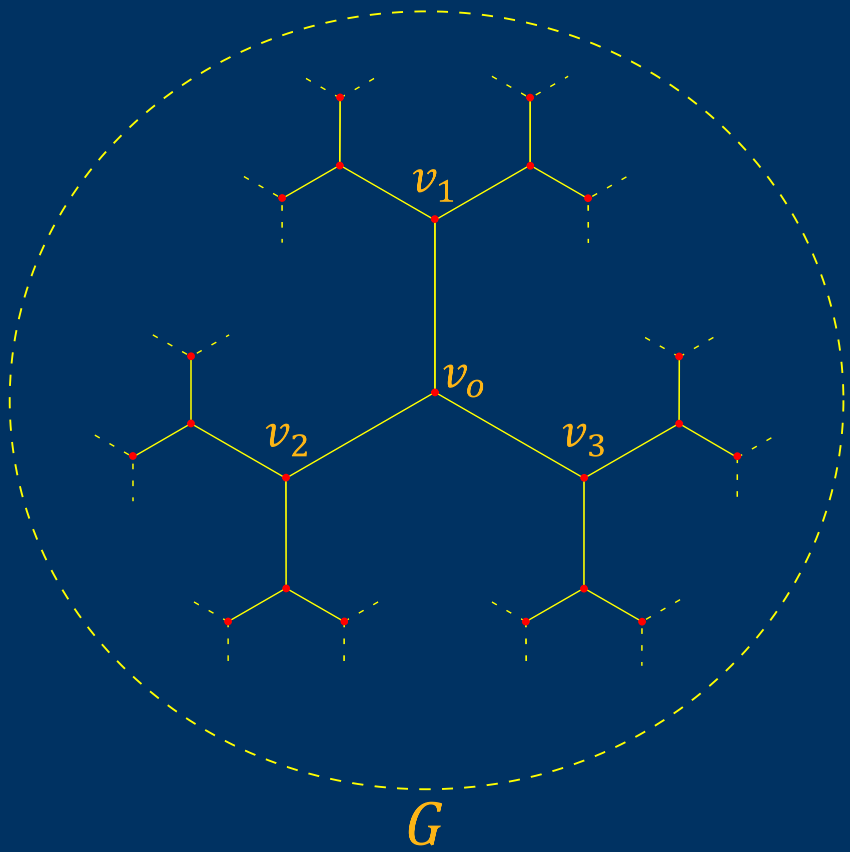
Backhausz and Szegedy

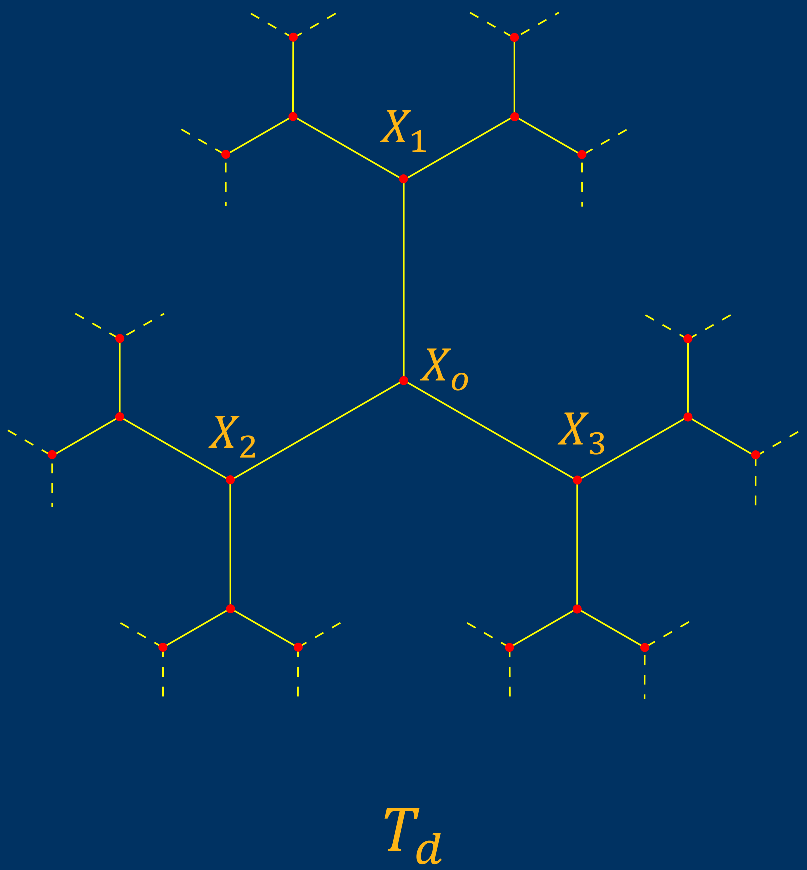
- For any $R, \epsilon > 0$, for large enough n, with probability at least $1 - \epsilon$, a random regular graph has the following property:

- For any eigenvector of the adjacency matrix, sample a vertex uniformly at random.

- The distribution of values in the R neighborhood of the vertex is at most ϵ in the weak topology from the distribution of the R-neighborhood in a multiple of the Gaussian wave.

Illustration of BS 19

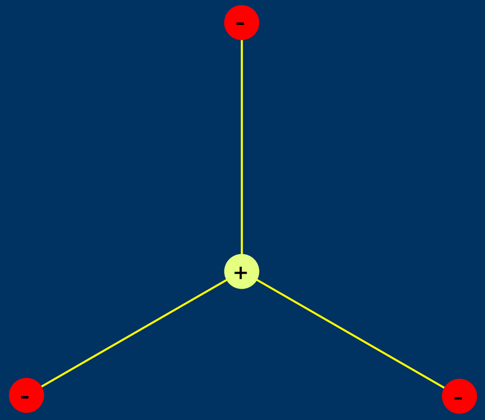




The distribution of $\sqrt{n}(v_o, v_1, v_2, v_3)$ when the central vertex is selected uniformly at random is close to the distribution of the Gaussian wave $\sigma \cdot (X_0, X_1, X_2, X_3)$ for some $\sigma \in [0,1]$.

Localization

- For negative λ, there is a constant probability that entries in the Gaussian wave correspond to a singleton nodal domain.



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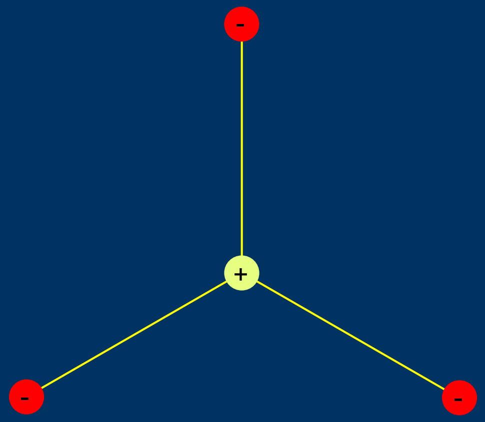
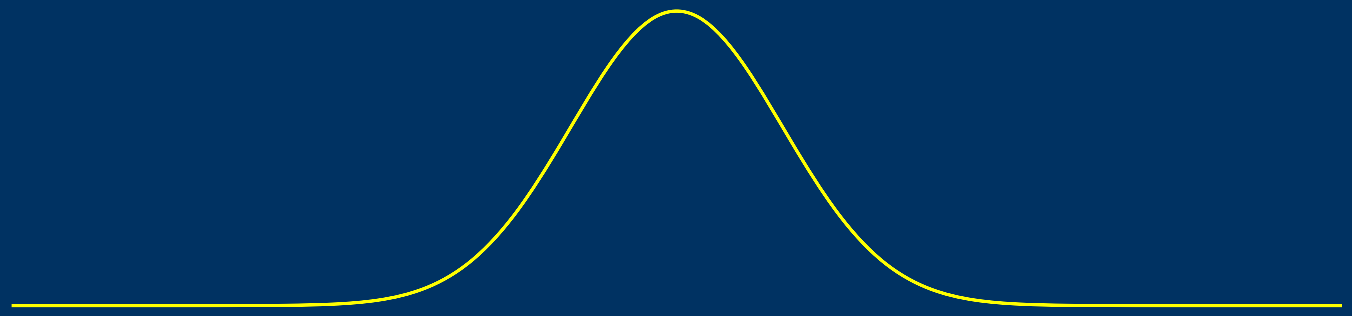


- However, the Gaussian wave our distribution is close to could have low variance (even variance 0). In this case, the eigenvector distribution's proximity to the Gaussian wave does not imply that there are many nodal domains.

- In this case, the eigenvector is **localized**.

Localization



If the eigenvector entries are close to a Gaussian with nonzero variance, then the proximity to the Gaussian wave implies many nodal domains

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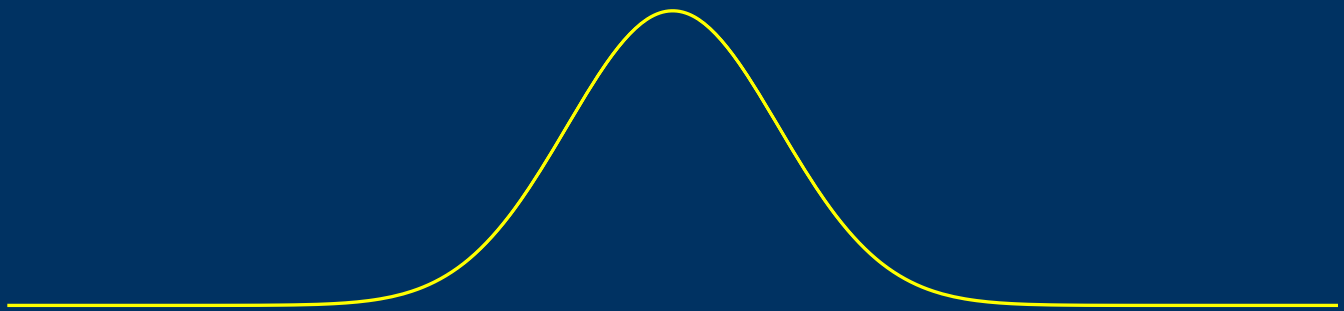
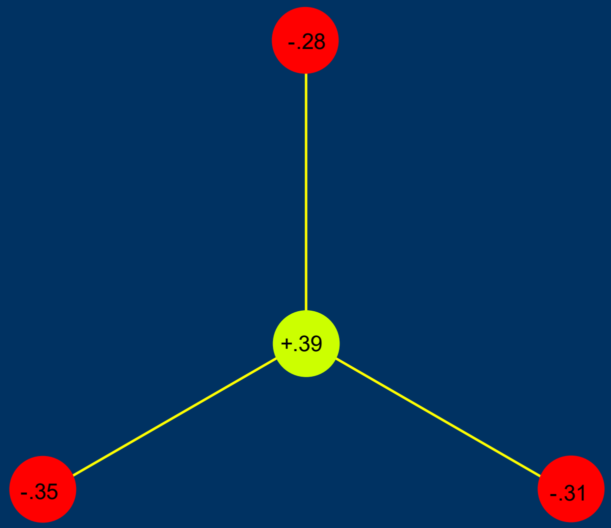


If the eigenvector entries are close to a Gaussian with nonzero variance, then the proximity to the Gaussian wave implies many nodal domains

If the entries are close to a Gaussian with variance close to zero, then there is no such implication, but we also know that most eigenvector entries are concentrated around 0.

Localization



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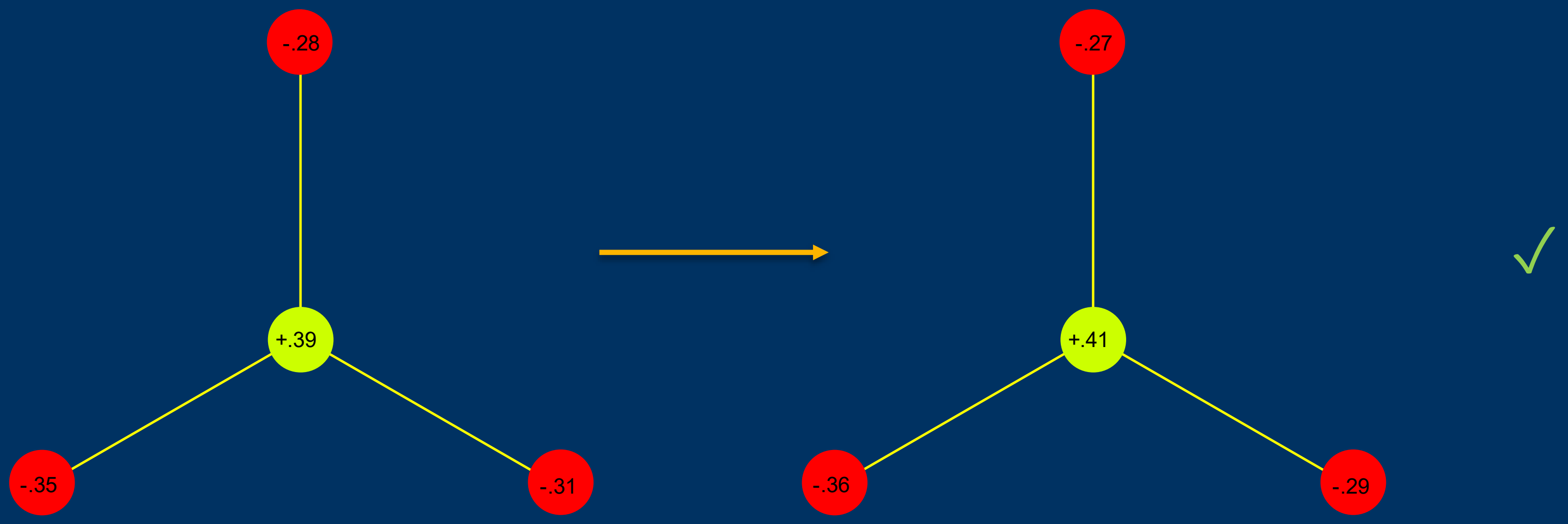

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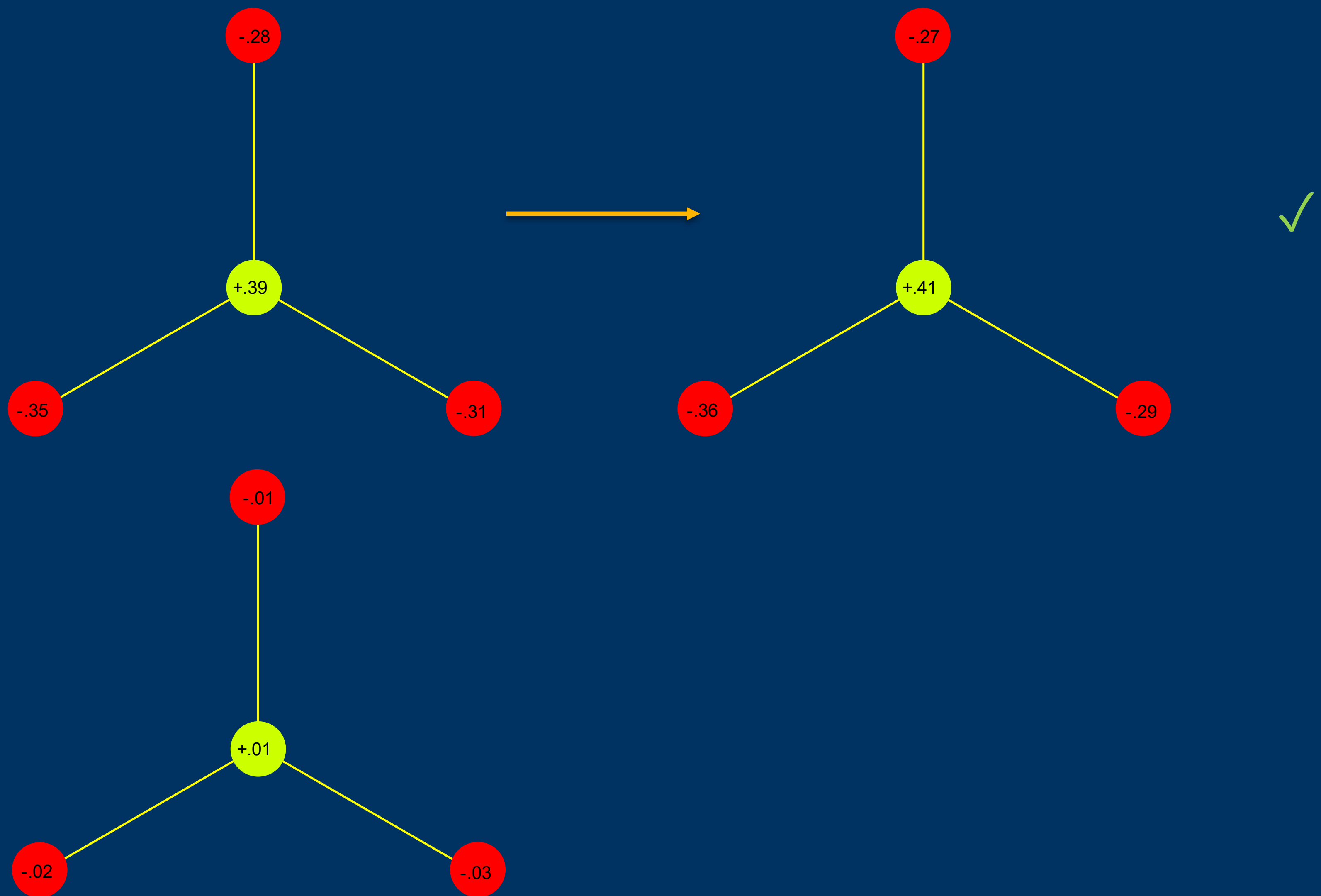

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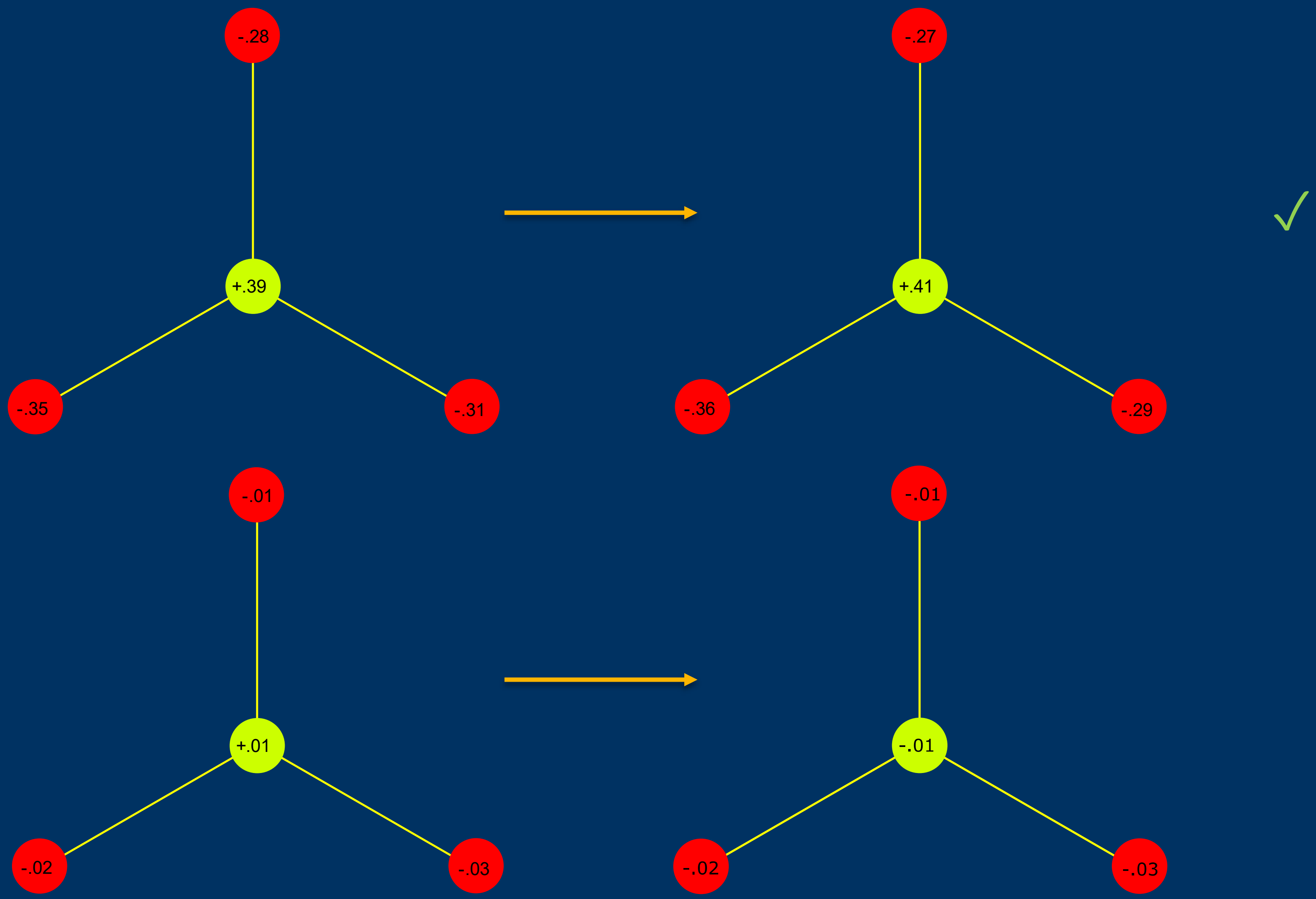

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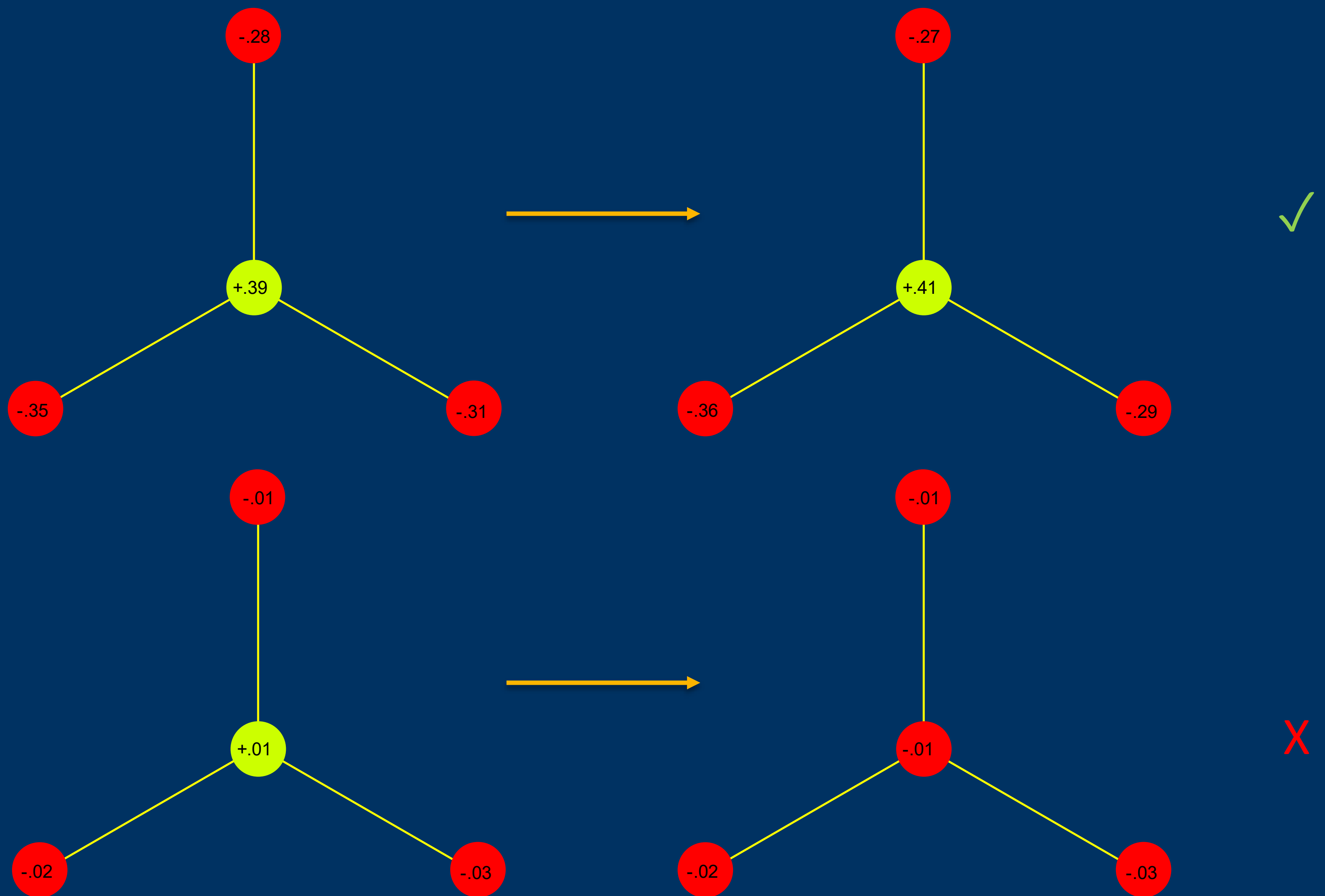

Outline

- We split into cases based on whether the eigenvector is **localized** or **delocalized** (whether the mass of the eigenvector is well spread or not).

- **Definition:** an eigenvector ψ is **delocalized** if for fixed $\epsilon, \delta > 0$,

$$|\{v \in V | \psi^2(v) \geq \epsilon/n\}| \geq \delta n.$$

- If the eigenvector is delocalized, we can use the proximity of an eigenvector of a random regular graph to a Gaussian distribution. ✓

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Green's Function Bound

- It remains to solve the problem when the eigenvector is localized.

- We use another connection between the infinite tree and the random regular graph.

- [Bauerschmidt-Huang-Yau 2019, Huang-Yau 2021$^+$] With high probability, the adjacency matrix A_G of a random regular graph G sampled from $G(n,d)$ is such that for any $z \in \mathbb{C}$ with $\Im(z) \geq \log^C n/n$

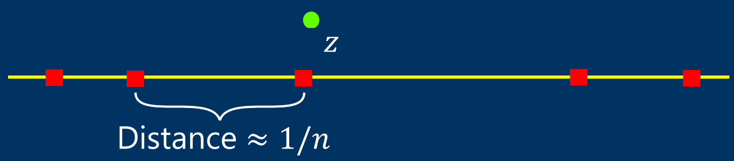
$$(z - A_G)^{-1}{}_{u,v} \approx \left(z - A_{T_d}\right)^{-1}{}_{u^*,v^*}$$

where $u^*, v^* \in T_d$ and the graph distance of u^* and v^* is the graph distance of u and v.

Function Bound Continued

- We want to use the Green's function to analyze the structure of one eigenvector. However, because we must have $\Im(z) \geq \log^C n/n$, we cannot separate one eigenvector from its closest $\log^C n$ neighbors.

z's Distance from the Real Line



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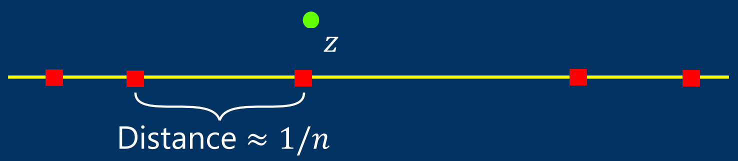

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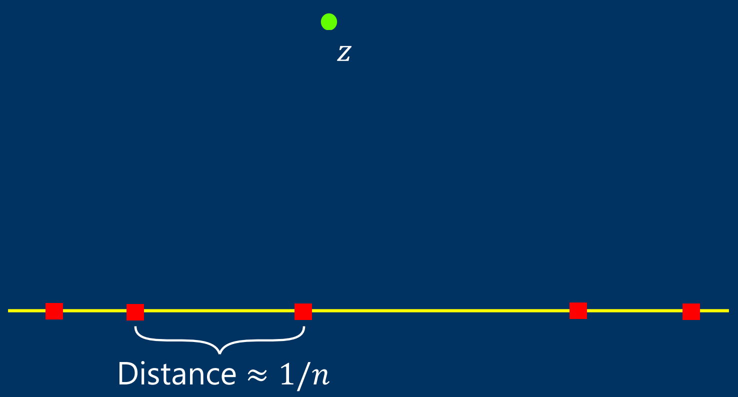

- If $\Im(z) \geq \log^C n/n$, then we cannot separate the desired eigenspace from the closest $\log^C n$ eigenspaces.

Infinity norm

- Nevertheless, the Green's function bound implies a nearly optimal bound on the ∞-norm.

- [Bauerschmidt-Huang-Yau '19, Huang-Yau '21+] Corollary: With high probability, any eigenvector ψ of the adjacency matrix of a $G(n,d)$ graph satisfies
$$\|\psi\|_\infty \le \log^{C/2} n/\sqrt{n}.$$
- Using this, then further analyzing properties of the localized vector, we can prove our result.

Proof for very negative eigenvalues

- Here we present a simpler version of our ideas, that only uses eigenvector delocalization, and no other properties of random regular graphs.

Proof for very negative eigenvalues continued

- Assume that ψ is an eigenvector of eigenvalue $\lambda \leq -d + 1 - \epsilon$.

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- As u has at most $d - 1$ neighbors such that $\psi(u)\psi(v) \leq 0$, it must be the case that there is some neighbor v_1 of u such that
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- By delocalization, there are many such starting vertices u, meaning there are many singleton nodal domains.

Small sets on random graphs

- With high probability, a $G(n, d)$ graph does not have eigenvalues below $-2\sqrt{d-1} - o_n(1)$, whereas this method works for $\lambda \leq -d + 1 - \epsilon$. Therefore, this is not useful for random regular graphs when $d \geq 5$.

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- To achieve our better bound, we work globally rather than locally.

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- Consider the set of vertices u such that $|\psi(u)| \geq \epsilon/\sqrt{n}$ for some small ϵ. Namely, there is nontrivial eigenvector mass on the vertex.

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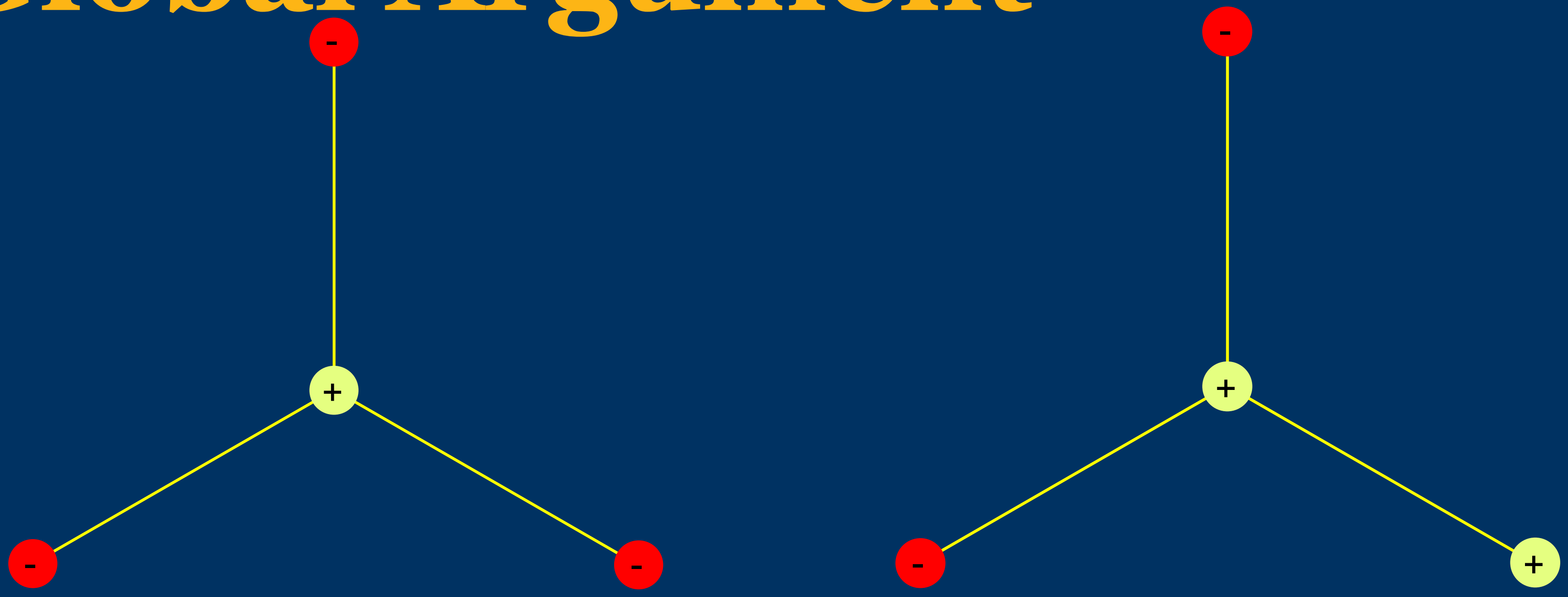
- Assume that none of these vertices are singleton nodal domains. **Then we can remove an edge neighboring each of these vertices while only decreasing the Rayleigh quotient**.

Global Argument



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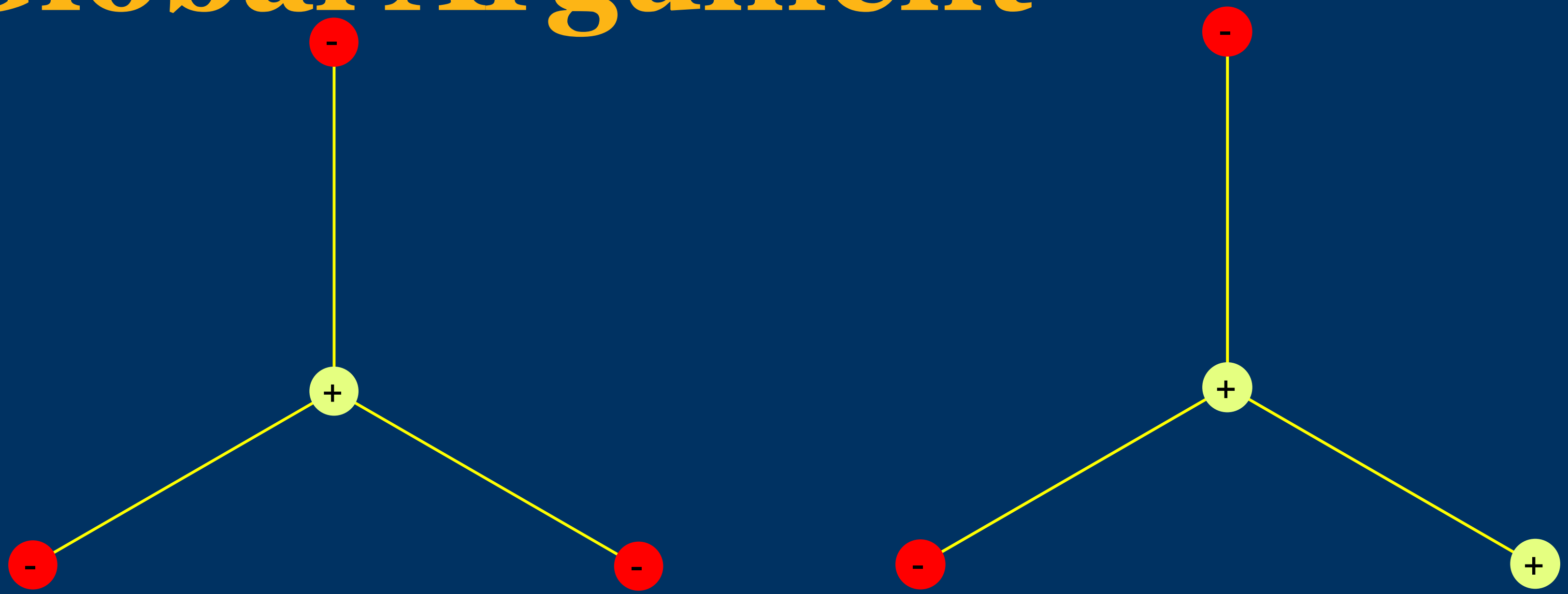

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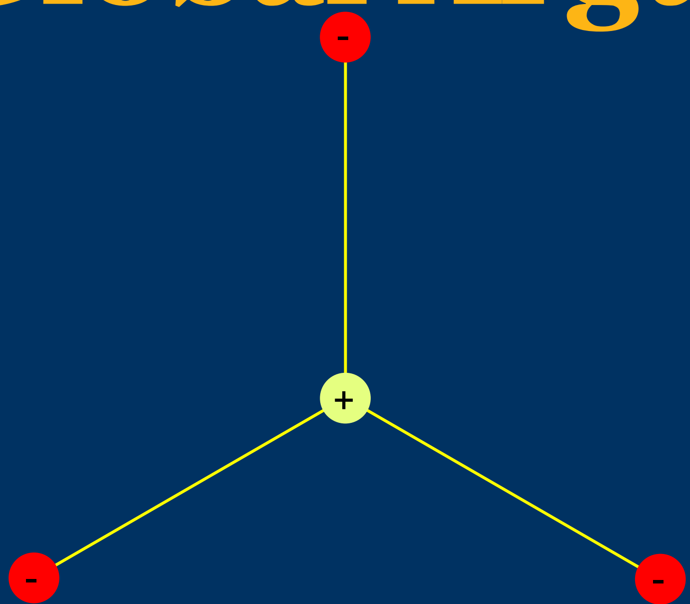

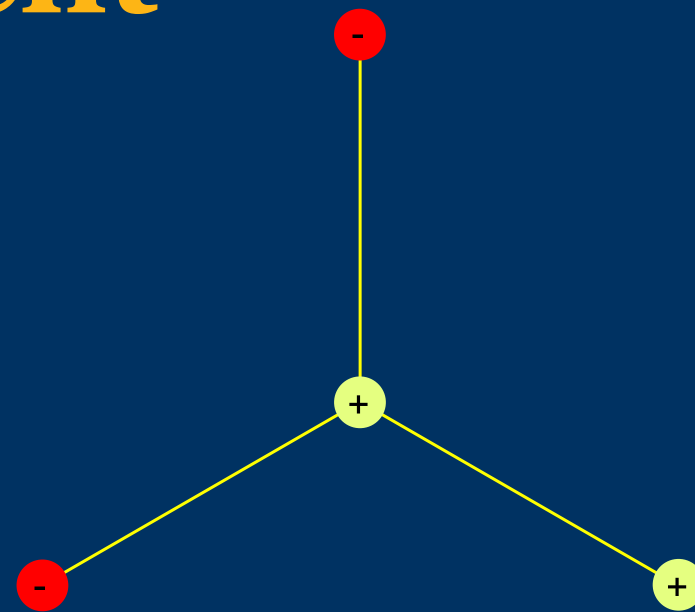

- $\psi^T A \psi = \sum_{u \sim v} \psi(u)\psi(v) = \lambda.$

- If u is a singleton nodal domain, then all terms corresponding to u in $\psi^T A \psi$ are negative.

- However, if u is not a singleton nodal domain, then there is at least one term that is positive.

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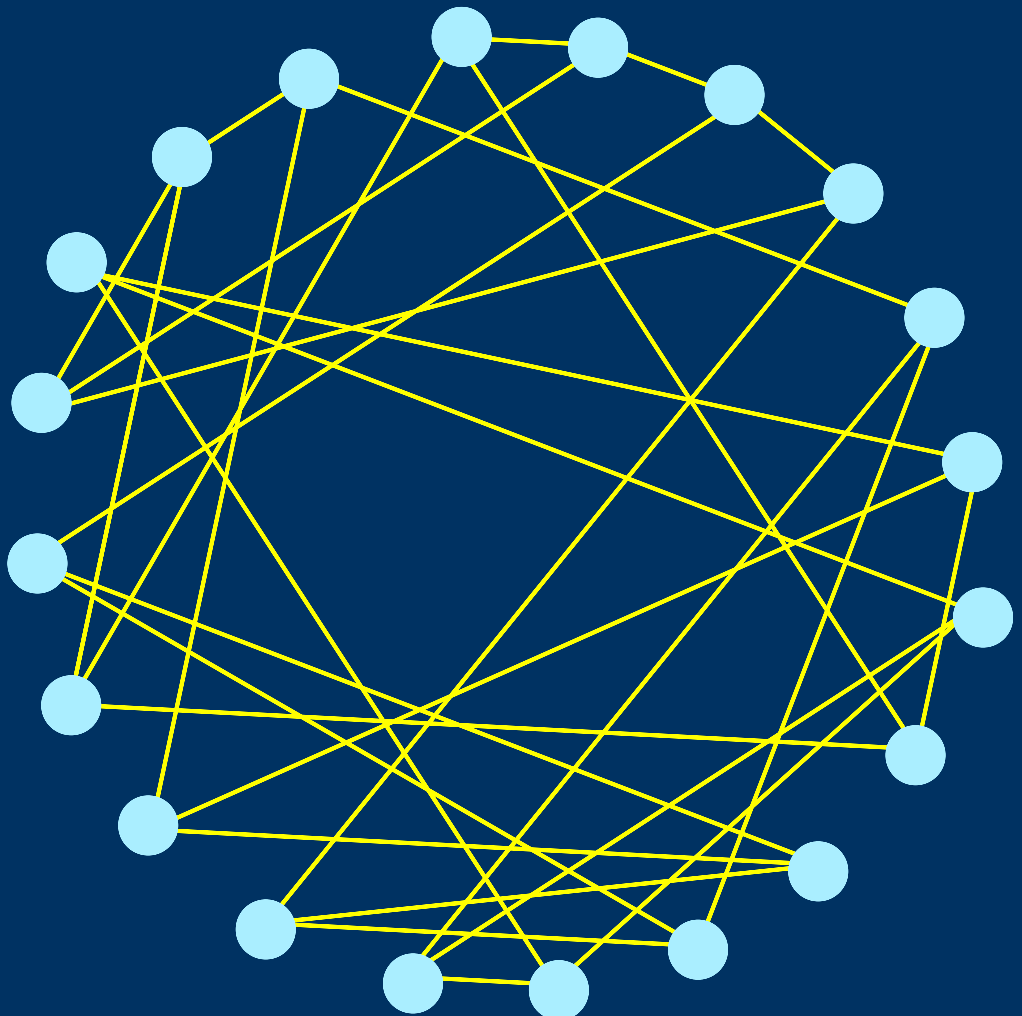
- We then delete all edges (u, v) such that $\psi(u)\psi(v) \geq 0$.
- As there are no singleton nodal domains, we have deleted at least one edge from each remaining vertex.
- We are left with a subgraph H of maximum degree $d - 1$ such that the vector projected onto this subgraph has Rayleigh quotient

$$\psi_H^T A_H \psi_H \leq \psi^T A \psi + \epsilon$$

for some small ϵ.

Process



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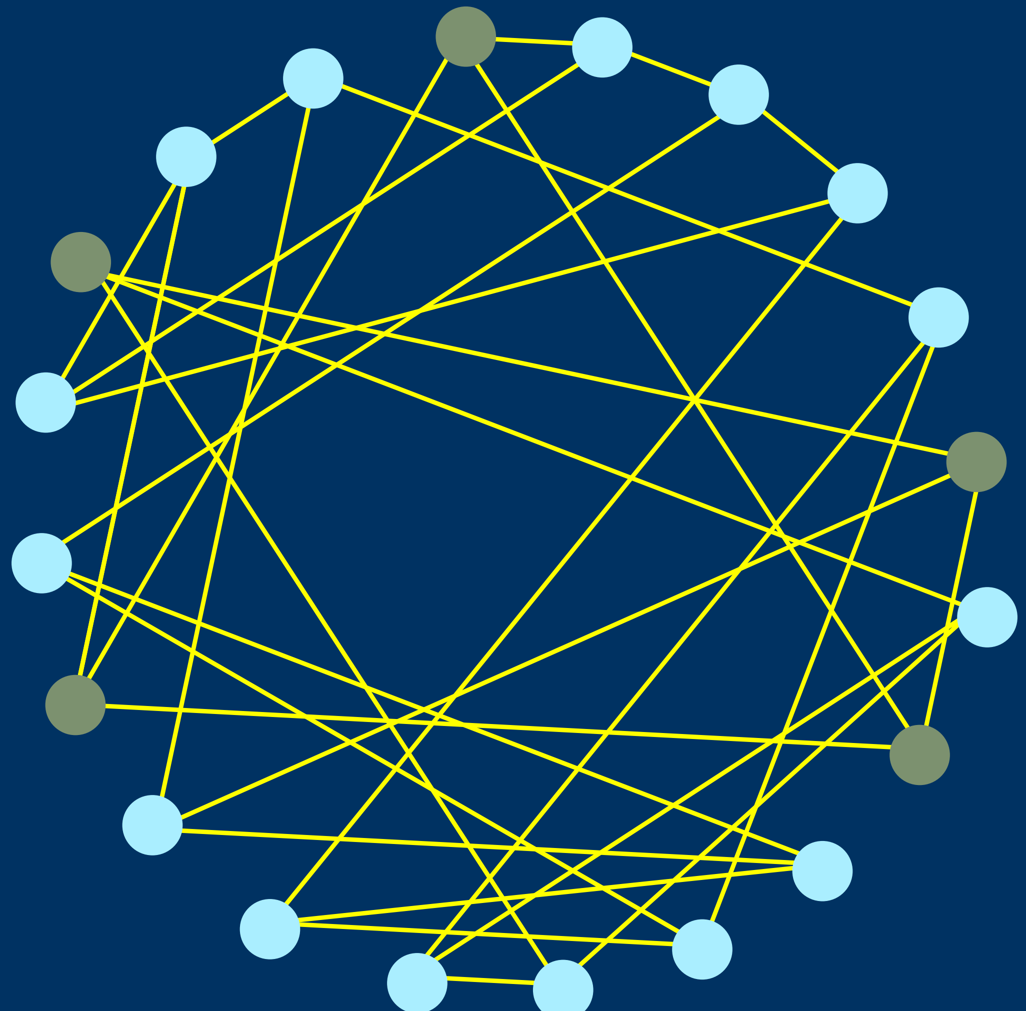

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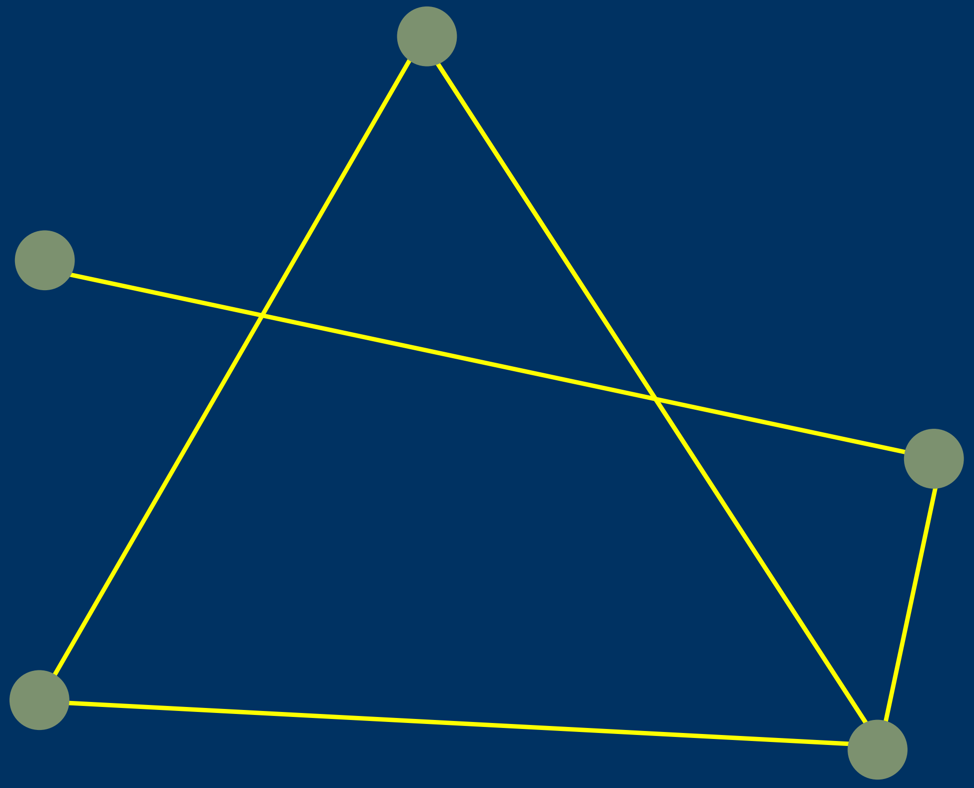

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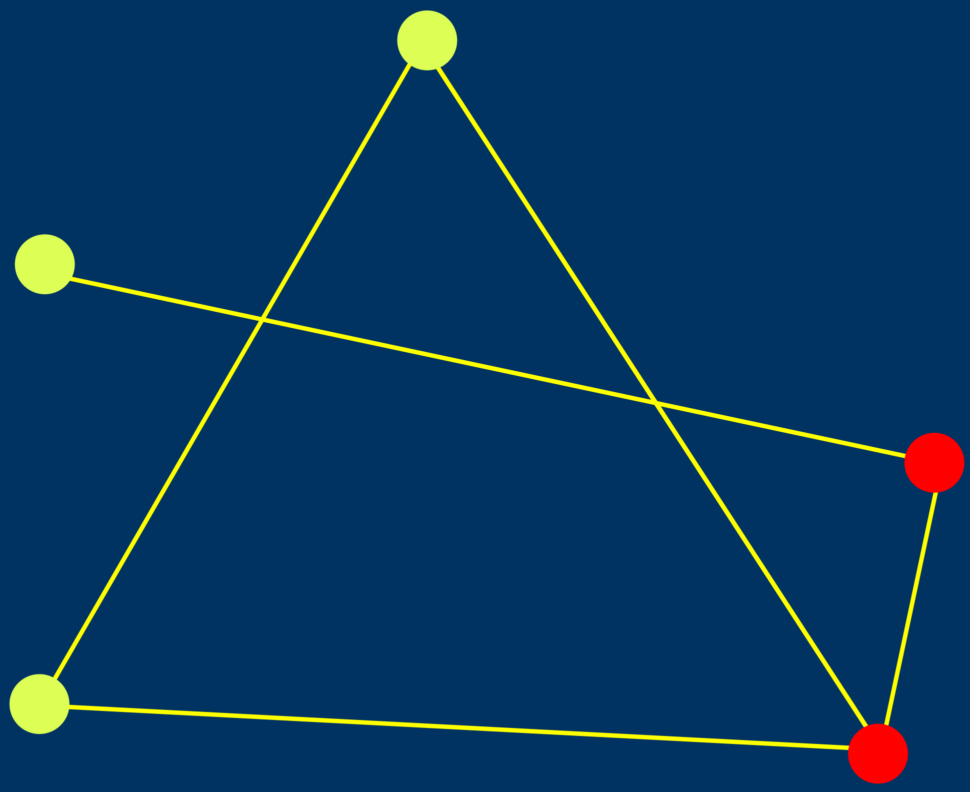

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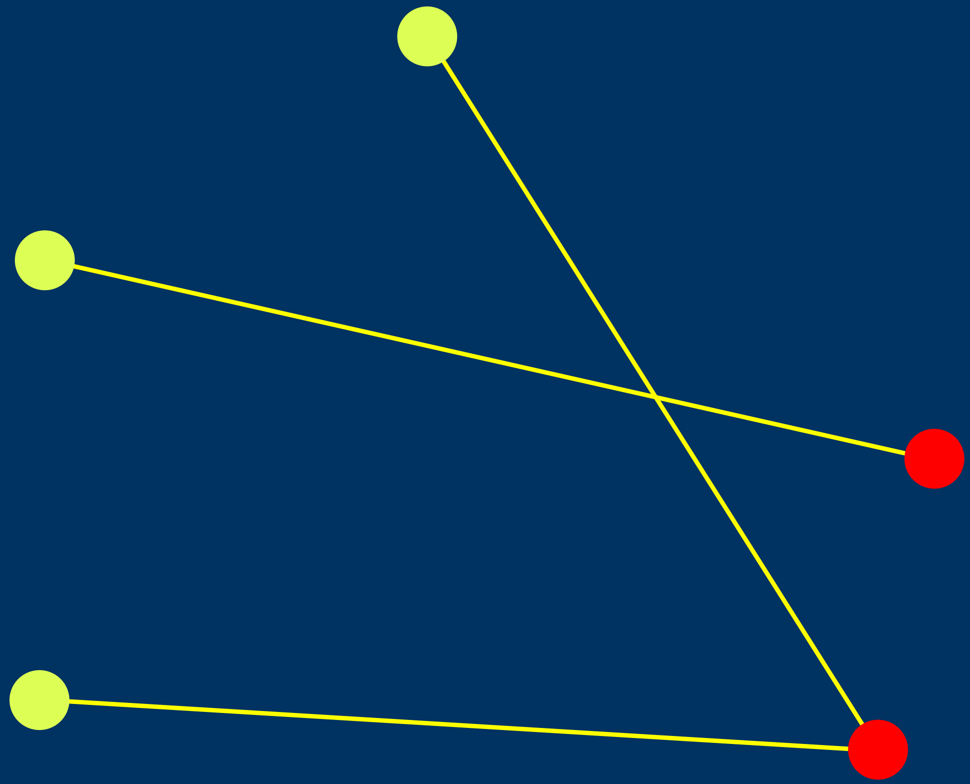

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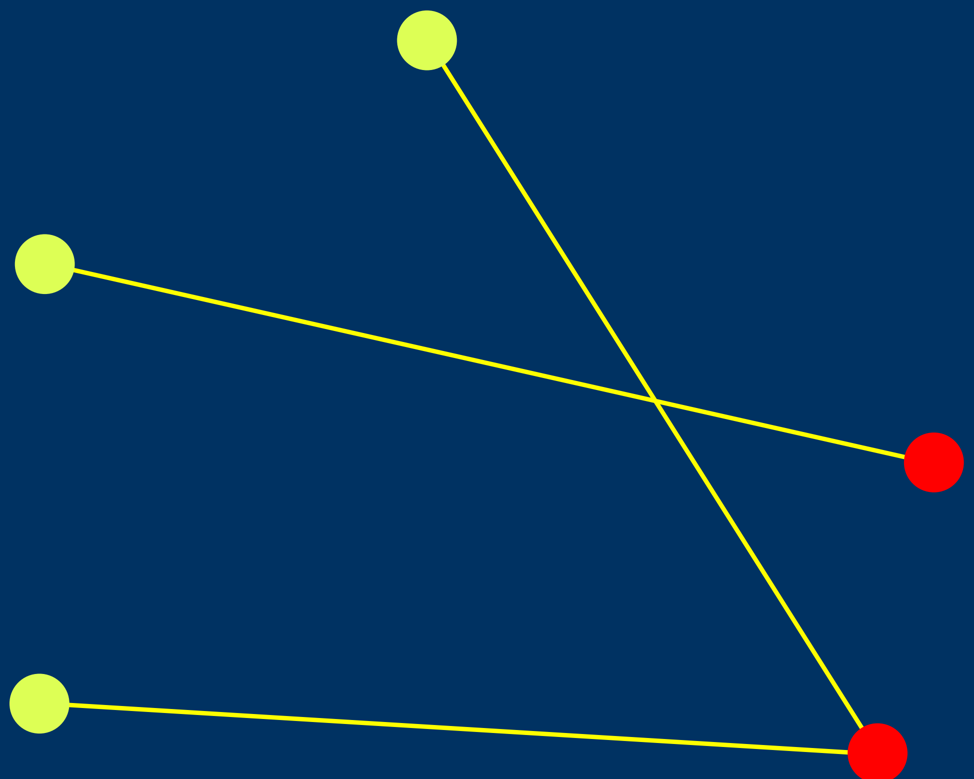

At no step did we significantly increase the Rayleigh quotient of ψ.
Therefore, $\psi_S^T A_S \psi_S \leq \lambda + \epsilon$.

This subgraph is **small** and **has maximum degree $\boldsymbol{d-1}$**.

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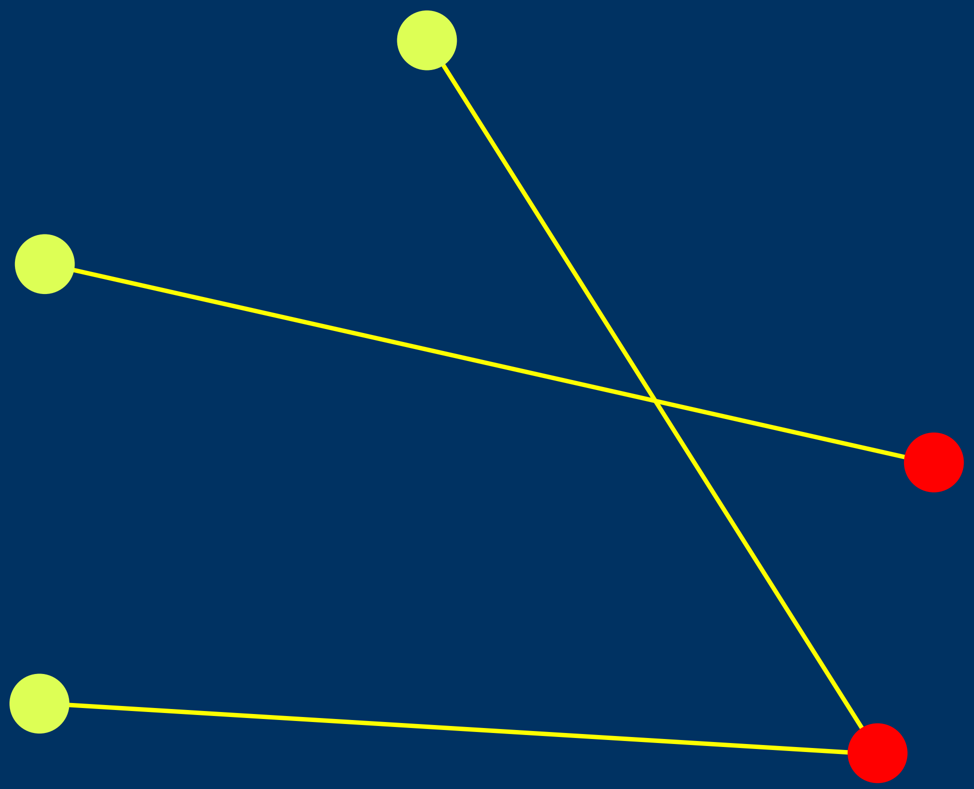
- **Claim**: Because the maximum degree is $d-1$ and the graph is close to tree-like, the spectral radius cannot be more than $2\sqrt{d-2} + \epsilon$.
- This can be proven through proving that the quadratic form on this tree-like graph is a convex combination of the quadratic forms of $(1+\delta)\psi A_T \psi$, where A_T is the adjacency matrix of a subtree.

Local Structure Continued



- The resulting graph is close to treelike and has maximum degree $d-1$. Therefore, the spectral radius is $\leq 2\sqrt{d-2}+\epsilon$

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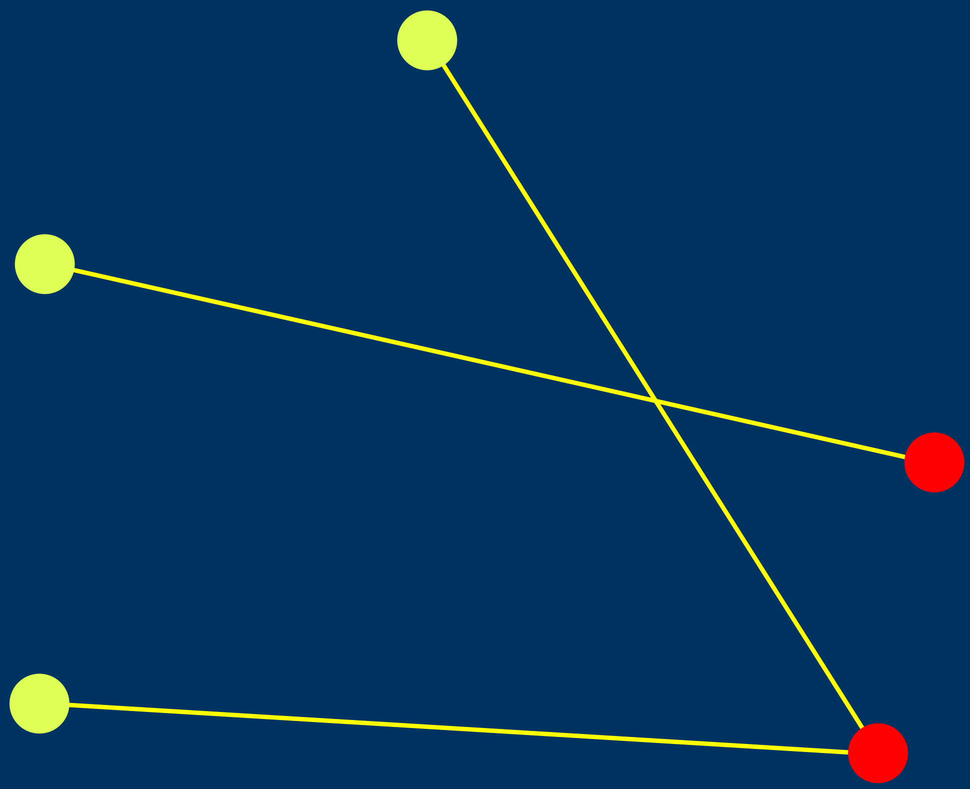

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- If λ is less than this than our assumption that there are few nodal domains must be false.

Recap

- As is the case with previous nodal domain results, we require bounds on the delocalization of our eigenvector, even though we prove a lower bound instead of an upper bound on the number of nodal domains.

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- We even do this in the localized case.

In the Future

- We want to improve the parameters of our theorem.

[Ganguly-M-Mohanty-Srivastava] Fix $d \geq 3$ and $\alpha > 0$. Then with probability $1 - o(1)$, every eigenvector of the adjacency matrix of a $G(n, d)$ sampled graph with eigenvalue $\lambda \leq -2\sqrt{d-2} - \alpha$ has $\Omega(n/\text{polylog}(n))$ nodal domains.

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- To improve the bound on λ, perhaps we could utilize the lack of nodal domains of different sizes besides just singletons, or find other “holes” in the spectrum.

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Claim: At least $\left(1 - \frac{5}{\sqrt{d}}\right)$ vertices lie in the largest negative and largest positive nodal domain.

In the Future

- Finally, it is worth mentioning that in the original Erdős-Rényi case, little is known about $G(n, p)$ for $p \leq n^{-1/20}$. Eldan, H. Huang and Rudelson suggested considering the critical level for connectivity $p = c\ \log n/n$ to see if for p at this level there are many nodal domains.

In the Future

- Finally, it is worth mentioning that in the original Erdős-Rényi case, little is known about $G(n, p)$ for $p \leq n^{-1/20}$. Eldan, H. Huang and Rudelson suggested considering the critical level for connectivity $p = c \log n/n$ to see if for p at this level there are many nodal domains.

- Linial suggested to consider the geometry of nodal domains. This includes classifying vertices by their distance to the boundary of the nodal domain.

Thank you!